Chapter 12

Vector Integral Calculus
12.1 Line integral

1. Curve in 3-space

(1) parametric equation
\[ C : x = x(t), y = y(t), z = z(t), a \leq t \leq b \]
initial point \((x(a), y(a), z(a))\)
final point \((x(b), y(b), z(b))\)
Ex. Curve: \(C_1 : x = 2 \cos t, y = 2 \sin t, z = 4, 0 \leq t \leq 2\pi \)
Curve: \(C_2 : x = 2 \cos t, y = 2 \sin t, z = 4, 0 \leq t \leq 4\pi \)

(2) Continuous
\[ X(t), Y(t), Z(t) \] are continuous on \(a \leq t \leq b\)
\[ C_1 : X = 2 \cos t, Y = 2 \sin t, Z = 4, 0 \leq t \leq 2\pi \]
\[ \therefore 2 \cos t, 2 \sin t, 4 \] continuous on \(0 \leq t \leq 2\pi\)
\[ \therefore C_1 \text{ is continuous} \]

(3) differentiable
\[ X(t), Y(t), Z(t) \] are differentiable

(4) Smooth \(\text{ (No sharp points or corners)}\)
\[ X'(t), Y'(t), Z'(t) \] are continuous on \(a \leq t \leq b\) and not all zero
\[ C_1 : X = 2 \cos t, Y = 2 \sin t, Z = 4, 0 \leq t \leq 2\pi \]
\[ X' = -2 \cos t, Y' = 2 \sin t, Z' = 0 \] are continuous on \(0 \leq t \leq 2\pi\)
\[ \Rightarrow \text{Curve } C_1 \text{ is smooth} \]

(5) Position vector of curve
\[ \vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, a \leq t \leq b \]
For smooth curve, \(\vec{R}(t) \neq 0\) and continuous

(6) Closed curve
Initial and terminal are the same
Ex: \(C_1 : X = 4 \cos t, Y = 4 \sin t, Z = 4, 0 \leq t \leq 3\pi \)
\[ C_2 : X = 4 \cos t, Y = 4 \sin t, Z = 4, 0 \leq t \leq 4\pi \]
\[ C_1 : \text{not closed curve} \]
\[ C_2 : \text{closed course} \]
2. Line Integrals of vector Field

(1) Definition
The line integral of vector field \( \vec{F}(x, y, z) \) over a smooth curve \( c \) is denoted
\[
\int_c \vec{F} \cdot d\vec{R}
\]
and is defined by
\[
\int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t) \, dt
\]
where \( \vec{R}(t) \) is position vector of a smooth curve \( c \) for \( a \leq t \leq b \), and \( \vec{F} \) is continuous at points on \( c \).

\[
\vec{F} \cdot d\vec{R} = \vec{F}(t) \cdot \frac{d\vec{R}}{dt} dt
\]

Ex: Evaluate \( \int_c \vec{F} \cdot d\vec{R} \)

Where \( \vec{F} = xi - yj + e^z \khat \)
\( c: x = t^3, y = -t, z = t; 0 \leq t \leq 1 \)

Sol: \( \vec{R}(t) = t^3 \hat{i} - t \hat{j} + t \hat{k}; 0 \leq t \leq 1 \)
\( \vec{F}(t) = t^3 \hat{i} - y \hat{j} + e^z \khat \)
\( = t^3 \hat{i} - (-t) \hat{j} + e^z \khat \)
\( = t^3 \hat{i} + t^2 \hat{j} + e^z \khat \)

\[
\therefore \int_c \vec{F} \cdot d\vec{R} = \int_0^1 \vec{F} \cdot \vec{R}' \, dt = \int_0^1 (t^3, t^2, e^z) \cdot (3t^2, -1, 1) \, dt
\]
\( = \int_0^1 (3t^5 - t^2 + e^z) \, dt \)
\( = \frac{5}{6} + e \)

(2) Piecewise smooth curve (or Path)
A curve where position vector \( \vec{R}(t) \) is continuous and tangent vector \( \vec{R}'(t) \) is continuous and different from zero vector at all but possibly a finite number of values of \( t \).

(3) line integral to piecewise smooth curve
\[
\int_c \vec{F} \cdot d\vec{R} = \int_{c_1} \vec{F} \cdot d\vec{R} + \int_{c_2} \vec{F} \cdot d\vec{R} + \ldots + \int_{c_n} \vec{F} \cdot d\vec{R}
\]
where \( c \) is a piecewise smooth curve, consisting of smooth curves
\( c_j : j = 1, 2, 3, \ldots, n \), and \( \vec{F} \) is a continuous vector field on \( c \).
(4) Physical Interpretation

Line integral of vector field \( \int_c \vec{F} \cdot d\vec{R} \) can be interpreted as the work done by a force \( \vec{F} \) in moving the particle over the path \( c \).

\( d\vec{R} \) : displacement of particle

\( \text{work} = dU = \int_c \vec{F} \cdot d\vec{R} \)

\[ U = \int dU = \int_c \vec{F} \cdot d\vec{R} \]

Ex: Given: Force \( \vec{F} = \vec{i} - \vec{j} + xyz\vec{k} \)

curve \( c : x=t, y=t^2, z=t; 0 \leq t \leq 1 \)

Find: work done by force \( \vec{F} \) in moving a particle along curve

Sol: work = \( \int_c \vec{F} \cdot d\vec{R} = \int_0^1 \vec{F} \cdot R'(t) \, dt \)

\( \vec{R}(t) = t\vec{i} - t^2\vec{j} + t\vec{k} \)

\( \vec{R}'(t) = \vec{i} - 2t\vec{j} + \vec{k} \)

\[ \vec{F}(x(t), y(t), z(t)) = \vec{i} - \vec{j} + xyz\vec{k} \]

\[ = \vec{i} + t^2\vec{j} - t^4\vec{k} \quad \text{on} \ c \]

\[ \therefore \int_0^1 \vec{F} \cdot \vec{R}'(t) \, dt = \int_0^1 (1 - 2t - t^4) \, dt = \frac{3}{10} \]

(5) Properties

i. \( \int_c (\vec{F} + \vec{G}) \cdot d\vec{R} = \int_c \vec{F} \cdot d\vec{R} + \int_c \vec{G} \cdot d\vec{R} \)

ii. \( \int_c (\alpha \vec{F}) \cdot d\vec{R} = \alpha \int_c \vec{F} \cdot d\vec{R} \quad \alpha : \text{any number} \)

iii. \( \int_c \vec{F} \cdot d\vec{R} = -\int_{-c} \vec{F} \cdot d\vec{R} \)

3. Line Integrals of scalar field

\[ \int_c f(x, y, z) \, dx + g(x, y, z) \, dy + h(x, y, z) \, dz \]

\[ = \int_a^b \left[ f(x(t), y(t), z(t)) \frac{dx}{dt} + g(x(t), y(t), z(t)) \frac{dy}{dt} + h(x(t), y(t), z(t)) \frac{dz}{dt} \right] \, dt \]

where \( f, g, \) and \( h \) are continuous functions on curve \( c \) having parametric functions \( x=x(t), y=y(t), z=z(t) \) for \( a \leq t \leq b \).

If

\[ \vec{F}(x, y, z) = f\vec{i} + g\vec{j} + h\vec{k} \quad \vec{R} = x\vec{i} + y\vec{j} + z\vec{k} \]
then
\[ \vec{F} \cdot d\vec{R} = f\,dx + g\,dy + h\,dz \]

so
\[ \int_c (f\,dx + g\,dy + h\,dz) = \int_c \vec{F} \cdot d\vec{R} \]

Another way of line integral of a vector field

4. Line integral wrt Arc length

(1) Definition
\[ \int_c f(x, y, z)\,ds = \int_a^b f(x(s), y(s), z(s))\,ds \]
\[ = \int_a^b f(x(t), y(t), z(t))\,ds\,dt \]

Where \( c \) is smooth curve and \( f(x, y, z) \) is real-valued function

If
\[ c = \vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad \text{for} \quad a \leq t \leq b \]

then arc length \( s(t) \) along \( c \) is

\[ s(t) = \int_a^t \left\|\frac{d\vec{R}}{d\zeta}\right\|\,d\zeta \]

\[ \therefore \frac{ds}{dt} = \left\|\frac{d\vec{R}}{dt}\right\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \]

Hence
\[ \int_c f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\,ds\,dt \]
\[ = \int_a^b f(x(t), y(t), z(t))\left\|\frac{d\vec{R}}{dt}\right\|\,dt \]
(2) Engineering Application

(i) mass of a wire

\[ \text{mass} = \int_c^b \delta(x, y, z) \frac{ds}{t} = \int_a^b \delta(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt \]

where \( \delta(x, y, z) \) is the density of the material

(ii) center of mass of the wire

\[
\begin{align*}
\bar{x} &= \frac{1}{m_c} \int x \delta(x, y, z) \, ds \\
\bar{y} &= \frac{1}{m_c} \int y \delta(x, y, z) \, ds \\
\bar{z} &= \frac{1}{m_c} \int z \delta(x, y, z) \, ds \\
m &= \int \delta(x, y, z) \, ds
\end{align*}
\]

12.2 Green’s Theorem

1. line integral along simple closed curves

\[ \oint_C \vec{F} \cdot d\vec{R} = \int_C f(x, y) \, dx - g(x, y) \, dy \]

(1) positive orientation of closed curve \( C \)

\[ c: x = x(t), \quad y = y(t), \quad a \leq t \leq b \quad \text{in the plane} \]

\((x(t), y(t))\) moves around \( C \) counterclockwise.

(2) simple curve

same point cannot be on the curve for different values of parameter.

(3) simple closed curve

A closed curve where initial and terminal point are the only point that coincide for different values of the parameter.

2. Green Theorem on the plane

(1) Definition
\[
\int_C \tilde{F} : \tilde{R} = \int_C f dx + g dy
\]
\[
= \iint_D \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA
\]

Where \( C \) is a simple closed positively oriented path

\( D = c \cup \text{interior of } c \)

\( \tilde{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j} \)

and \( f, g, \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} \) are continuous on \( D \)

(2) Applications

(i) Convert a difficult integration into an easier one

EX: Force \( \tilde{F} = (y - x^2 e^x)\hat{i} + (\cos y^2 - x)\hat{j} \)

\[
f(x, y) = y - x^2 e^x
\]
\[
g(x, y) = \cos y^2 - x
\]
\[
\frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = -1
\]

From Greens Theorem

\[
\oint = \tilde{F} : d\tilde{R} = \iint_D \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA = -2 \iint_D dA = -4
\]

(ii) Evaluate general line integral

\[
I : \oint_0^2 x \cos(2y)dx - 2x^2 \sin(2y)dy
\]

\( C : \text{any positively oriented path} \)

\[
\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0
\]

\[
\therefore I = \iint_D \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA = 0
\]

12.3 Independence of path and potential Theory

1. Conservative Vector Field

(1) Definition

Vector field \( \tilde{F} \) is called conservative if \( \tilde{F} = \nabla \phi \)

Scalar field \( \phi \) is called a potential (function) for \( \tilde{F} \)

(2) Line integral
\[ \vec{F} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} \]
\[ \vec{R}' = \frac{dx}{dr} \hat{i} + \frac{dy}{dt} \hat{j} \]
\[ \therefore \vec{F}(x(t), y(t)) \cdot \vec{R}'(t) = \frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} = \frac{d\phi(x(t), y(t))}{dt} \]

\[ \Rightarrow \int_C \vec{F} \cdot d\vec{R} = \int_a^b F_x X(t)Y(t) \]
\[ \int_R (t) dt = \int_a^b \frac{d\phi}{dt} dt = \int_a^b \phi = \phi (X(t), Y(t)) \int_b^a = \phi \]

\[ (X(b), Y(b)) \phi (X(a), Y(a)) \]

2 Independence of path

(1) Definition

\[ \int_C \vec{F} \cdot d\vec{R} \] is independent of path in D if the integral has the same value over any path in D for Pt, Po to Pt, P1

(2) Line integral closed path if \[ \vec{F} = \nabla \phi \]

then \[ \int_C \vec{F} \cdot d\vec{R} \] is independent of path in D. If C is a closed path in D, then

\[ \int_C \vec{F} \cdot d\vec{R} = 0 \]

(3) Criterion of a conservative field vector field

\[ \vec{F} = f(x, y) \hat{i} + g(x, y) \hat{j} \] is
conservativers on a region R if and only if, at \((x,y)\) in \(R\)

\[
\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}
\]

\[
\begin{align*}
F & \quad d \quad R \\
\int_R &= f(x,y)dx + g(x,y)dy \\
\phi &= \int_{\partial R} = d \phi
\end{align*}
\]

\[
= \frac{\partial y}{\partial x} dx = \frac{\partial \phi}{\partial y} dy
\]

\[
\therefore \frac{\partial \phi}{\partial x} = f, \quad \frac{\partial \phi}{\partial y} = g
\]

\[
\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial \phi}{\partial y \partial x} = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}
\]

12-4 Surface Integrals

I surfaces (having area but no volume)

(1) parameter equation

\(X = X(u,v), Y = Y(u,v), Z = Z(u,v)\)

\(U, V\) two independent variables

Ex.

1. cylinder

\[
X^2 + Y^2 = a^2, -1 \leq z \leq 1,
\]

\(r(u, v) = a \cos u + a \sin v + v\)
\[0 \leq u \leq 2\pi, -1 \leq v \leq 1\]

(2) sphere
\[x^2 + y^2 + z^2 = a^2\]
\[r(u,v) = a \cos v \cos u \hat{i} + a \cos v \sin u \hat{j} + a \sin v \hat{k}\]

\[0 \leq u \leq 2\pi, \frac{\pi}{2} \leq v \leq \frac{\pi}{2}\]

(3) cone
\[z = \sqrt{x^2 + y^2} \quad 0 \leq z \leq h\]
\[r(u,v) = u \cos v \hat{i} + u \cos v \hat{j} + u \hat{k}\]

\[0 \leq u \leq H, 0 \leq v \leq 2\pi\]

(ii) \(Z=S(X,Y)\)

Lows of pts \((x, y, s(x, y))\)

Ex. Surface
\[z = \sqrt{4 - x^2 - y^2} \quad x^2 + y^2 \leq 4, \quad z \leq 0\]
Sol: square the equation
\[x^2 + y^2 + z^2 = 4\]
\[\Rightarrow\] hemisphere of radius 2 with
origin \((0,0,0)\)

Ex. Surface
\[z = \sqrt{x^2 - y^2} \quad x^2 + y^2 \leq 8\]
(2) position rector of a surface \( Z = S(X, Y) \)

\[
\begin{align*}
\mathbf{R} & \rightarrow (X_1, Y_1) = X + i + Y + j + S(x, y) \\
& \rightarrow k
\end{align*}
\]

(3) simple surface

The position rector \( R \) \((x, y)\) doesn’t return to the same point for different \((x, y)\).

\[
\begin{align*}
R & \rightarrow (X_1, Y_1) = R \rightarrow (X_2, Y_2) = X_1 = X_2 \quad Y_1 = Y_2
\end{align*}
\]

(4) Normal rector to a surface

A rector orthogonal to each rector on the tangent plane at Po

Write \( q(x, y, z) = s(x, y) \cdot z \)

So \( z = s(x, y) \) is level surface \( \varphi(x, y, z) = 0 \)

Gradient of \( \varphi \)

\[
\begin{align*}
\nabla \varphi &= \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k} \\
&= \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} - \frac{\partial \varphi}{\partial z} \mathbf{k} \\
&= N \rightarrow \text{ normal}
\end{align*}
\]

(5) smooth surface

A surface \( \sum \) is simple and has a continuous, nonzero normal rector at every pt.

(6) piecenenis smooth
A surface consists of a finite number of smooth surfaces.

Ex. Sphere surface : smooth
Surface area : pieconise smooth
(7)surface area

The area of a smooth surface \( \sum \) given by \( z : s( x, y ) \) is given by \( A( \sum ) = \)

\[
\iint_0 \sqrt{1 + s x^2 + sy^2} \, dx \, dy
\]

\[
= \iint_0 || \frac{x}{x} \rightarrow (x, y) || \, da
\]

= Double integral of the length of the normal vector.

2. Surface integral of a function of three variables
Let \( f(x,y,z) \) be a function of three variables defined over a region of space containing surface \( \sum \) then the surface integral of \( f(x,y,z) \) over a smooth surface \( \sum \) is found by

\[
\iint f(x,y,z) \, d\delta
\]

\[
= \iint f(x,y,s(x,y)) \sqrt{1 + sx^2 + sy^2} \, dA
\]

\[
= \iint f(x,y,z(x,y)) \sqrt{1 + zx^2 + zy^2} \, dA
\]

3. Some use of surface integrals
1. Surface area
If \( f(x,y,z)=1 \)

Then \( \iint d\delta = \iint \sqrt{1 + sx^2 + sy^2} \, dA = A(\sum) \)

2. Mass and cost of mass of a shell

\[
m = \iiint \delta(x,y,z) \, d\sigma = \int \int \delta(x,y,\delta(x,y)) \sqrt{1 + sx^2 + sy^2} \, dA
\]

\[
X = \frac{1}{m} \int \int x \delta(x,y,z) \, d\sigma
\]

\[
Y = \frac{1}{m} \int \int y \delta(x,y,z) \, d\sigma
\]
12.6 vector forms of green’s theorem

1. Green’s theorem in 2-space

\[ \oint (f dx + g dy) = \iint \left( \frac{\partial g}{\partial z} - \frac{\partial f}{\partial y} \right) dA \]

\( \forall F = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \)

(1) let \( F = gi - fj \)

\[ \oint f dx + g dy = \oint f \frac{dx}{ds} + g \frac{dy}{ds} \, ds = \oint F N ds \]

(2) let \( F(x,y,z)=f(x,y)i+g(x,y)j+0k \)

\[ \forall \times F = \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] K \]

curl \( F = (J \times F) \circ K = \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] \)

\[ \vec{F} \, ds = \left[ f \, i + g \, j \right] \left[ \frac{dx}{ds} \, i + \frac{dy}{ds} \, j \right] \, ds = f \, dx + g \, dy \]

so \( \oint f \, dx + g \, dy = \oint \vec{F} \cdot \vec{N} \, ds \)

Hence \[ \oint \vec{F} \cdot \vec{N} \, ds = \iiint_D \left( \nabla \cdot \vec{F} \right) \, K \, dA \]

2. Green Theorem in 3-space

(1) Gauss’s integral theorem

close curve \( C \) => close surface \( \sum \) lore

integral -> surface ontgray

\[ \int \int \int_D \vec{F} \cdot \vec{n} \, dV = \int \int \int_D \nabla \times \vec{F} \, dV \]

(3) stokes integral theorem

closed curve \( C \) -> a curve in 3 space flat surface \( D \) in xy plane -> a surface

\[ \sum \text{ with unit normal } \vec{I} \]

\[ \oint_c F \, ds = \int_D \left( \nabla \times \vec{F} \right) \cdot \vec{N} \, ds \]

\[ \Rightarrow \int_C \vec{F} \cdot \vec{d} = \int_a^b \vec{F} \cdot \vec{X(t)} \cdot \vec{Y(t)} \cdot \vec{R} \cdot (t) \, dt \]
\[
\int_{a}^{b} \frac{d\phi}{dt} dt = \int_{a}^{b} d\phi = \phi(X(t),Y(t)) \int_{b}^{a} = \phi(X(b),Y(b)) \phi
\]

\(( X(a), Y(a) )\)

2 Independence of path

(1) Definition

\[
\int_{C} F \rightarrow d \rightarrow R \rightarrow \text{B independent of path in D if the integral has the same value over any path in D for } P_{t}, P_{0} \text{ to } P_{t}, P_{1}
\]

(2) line integral closed path if \( \nabla \phi \rightarrow \)

then \(\int_{C} F \rightarrow d \rightarrow R \rightarrow \text{is independent of path in D. If } C \text{ is a closed path in D, then} \)

\(\int_{C} F \rightarrow d \rightarrow R \rightarrow = 0\)

(3) Criterion of a conservative field vector field

\(\rightarrow F \rightarrow = f(x,y) i + g(x,y) j \rightarrow \text{is conservative on a region } R \text{ if and only if, at } (x,y) \text{ in } R \)

\(\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}\)

\(\rightarrow F \rightarrow d \rightarrow R \rightarrow = f(x,y)dx + g(x,y)dy\)

\(= d\phi\)
\[ \frac{\partial y}{\partial x} \, dx = \frac{\partial \varphi}{\partial y} \, dy \]

\[ \therefore \frac{\partial \varphi}{\partial x} = f, \quad \frac{\partial \varphi}{\partial y} = g \]

\[ \frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial \varphi}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \]

12-4 Surface Integrals

I surfaces (having area but no volume)

(1) parametric equation

\( X = X(u, v), \ Y = Y(u, v), \ Z = Z(u, v) \)

\( U, V \) two independent variables

Ex.

(1) cylinder

\[ X^2 + Y^2 = a^2, \quad -1 \leq z \leq 1, \]

\[ r(u, v) = a \cos u \overset{i}{\rightarrow} +a \sin u \overset{j}{\rightarrow} +v \overset{k}{\rightarrow} \]

\[ 0 \leq u \leq 2\pi, \quad -1 \leq v \leq 1 \]

(2) sphere

\[ x^2 + y^2 + z^2 = a^2 \]

\[ r(u, v) = a \cos v \cos u \overset{i}{\rightarrow} +a \cos v \sin u \overset{j}{\rightarrow} \]

\[ u \overset{j}{\rightarrow} +a \sin v \overset{k}{\rightarrow} \]
\[ 0 \leq u \leq 2\pi, \quad \frac{N}{2} \leq v \leq \frac{\pi}{2} \]

(3) cone

\[ z = \sqrt{x^2 + y^2} \quad 0 \leq z \leq h \]

\[ r(u, v) = u \cos v \rightarrow i + u \cos \rightarrow j + u \rightarrow k \]

\[ 0 \leq u \leq H, 0 \leq v \leq 2\pi \]

(ii) \( Z = S(X, Y) \)

Lows of pts \((x, y, s(x, y))\)

Ex. Surface

\[ z = \sqrt{4 - x^2 - y^2} \quad x^2 + y^2 \leq 4, \quad z \leq 0 \]

Sol: square the equation

\[ \Rightarrow x^2 + y^2 + z^2 = 4 \]

\[ \Rightarrow \text{hemisphere of radius} \ 2 \ \text{with origin}(0,0,0) \]

Ex. Surface

\[ z = \sqrt{x^2 - y^2} \quad x^2 + y^2 \leq 8 \]

(2) position vector of a surface \( Z = S(X, Y) \)

\[ \mathbf{R} \rightarrow (X, Y) = X \rightarrow \mathbf{i} + Y \rightarrow \mathbf{j} + S(x, y) \rightarrow \mathbf{k} \]
(3) simple surface

The position vector $\mathbf{R}(x, y)$ does not return to the same point for different $(x, y)$.

$\mathbf{R}(x, y) = (x_1, y_1)\Rightarrow x_1 = x_2 \land y_1 = y_2$

(4) Normal vector $\mathbf{N}$ to a surface

A vector orthogonal to each vector on the tangent plane at $P_0$.

Write $q(x, y, z) = s(x, y) \cdot z$

So $z = s(x, y)$ is level surface $\varphi(x, y, z) = 0$

Gradient of $\varphi$

$\nabla \varphi = \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} - \frac{\partial \varphi}{\partial z} \hat{k}$

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$\mathbf{N} = \nabla \varphi = \text{normal}$

(5) Smooth surface

A surface $\sum$ is simple and has a continuous, nonzero normal vector at every point.

(6) Piecewise smooth

A surface consists of a finite number of smooth surfaces.

Ex. Sphere surface: smooth
Surface a cube: piecwise smooth

(7) Surface area

The area of a smooth surface $\sum$ given by $z = s(x, y)$ is given by

$$\int \int_0 \sqrt{1 + s_x^2 + s_y^2} \, dx \, dy$$
$$= \int \int_0 \| \frac{\partial}{\partial x} \rightarrow (x, y) \| da$$

= Double integral of the length of the normal vector.