CHAPTER 9

System of Linear Differential Equations

9.1 Theory of system of linear $1^{st}-order$ D.Es.

1. General $1^{st}-order$ linear system.

$$\begin{cases} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + g_1(t) \\ x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + g_2(t) \\ \vdots \\ x_{n1}'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + g_n(t) \end{cases}$$

In matrix form

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

or $X'(t) = A(t)X(t) + G(t)$

(1) Homogeneous system if $G(t) = 0$ for all t.

$X'(t) = A(t)X(t)$

(2) Nonhomogeneous system if $G(t) \neq 0$ for at least some t.

$X'(t) = A(t)X(t) + G(t)$

EX: 2×2 system $\begin{cases} x_1' = 3x_1 + 3x_2 + 8 \\ x_2' = x_1 + 5x_2 + 4e^{3t} \end{cases}$

In matrix form

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 8 \\ 4e^{3t} \end{bmatrix}$$ solution = ?

(3) Initial value problem

System D.E. $x'(t) = A(t)X(t) + G(t)$

Initial condition $x(t_0) = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_3^0 \end{bmatrix} = x^0$

Analogous to $1^{st}-order$ initial value problem.

$$x' = ax + g \; ; x(t_0) = x_0$$

2. Theory of homogeneous system $X' = AX$

(1) Linear combination of solutions

Let $\Phi_1, \Phi_2, \cdots, \Phi_k$ be solutions of $X' = AX$. Then

$C_1\Phi_1 + C_2\Phi_2 + \cdots + C_k\Phi_k$ is also a solution of $X' = AX$ where

$C_1, C_2, \cdots, C_k$ are numbers.

(2) Linear Dependence/Independence of solutions.

Solutions $\Phi_1, \Phi_2, \cdots, \Phi_r$ of $X' = AX$, defined on interval I, are linearly dependent if one solution is a linear combination of the others, otherwise are linear independent.

(3) Test for Linear Independent

suppose that

$$\Phi_1(t) = \begin{bmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{bmatrix}, \Phi_2(t) = \begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{bmatrix}, \cdots, \Phi_n(t) = \begin{bmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{bmatrix}$$

are solutions of $X' = AX$ in an open interval I. Let to be any number in I.

Then

(i) $\Phi_1, \Phi_2, \cdots, \Phi_n$ are linear independent on I iff

$\Phi_1(t_0), \Phi_2(t_0), \cdots, \Phi_n(t_0)$ are linear independent when considered as vectors in R^n.

(ii) $\Phi_1(t), \Phi_2(t), \cdots, \Phi_n(t)$ are linear independent on I.

iff

$$\begin{vmatrix} \phi_{11}(t_0) & \phi_{12}(t_0) & \cdots & \phi_{1n}(t_0) \\ \phi_{21}(t_0) & \phi_{21}(t_0) & \cdots & \phi_{n2}(t_0) \\ \vdots & & & \vdots \\ \phi_{n1}(t_0) & \phi_{n2}(t_0) & \cdots & \phi_{nn}(t_0) \end{vmatrix} \neq 0$$

$$C_1\phi_1(t_0) + C_2\phi_2(t_0) + \cdots + C_n\phi_n(t_0) = 0$$

$$C_1 = C_2 = \cdots = C_n = 0$$

$$\begin{bmatrix} \phi_{11}(t_0) & \phi_{12}(t_0) & \cdots & \phi_{1n}(t_0) \\ \phi_{21}(t_0) & \phi_{21}(t_0) & \cdots & \phi_{n2}(t_0) \\ \vdots & & & \vdots \\ \phi_{n1}(t_0) & \phi_{n2}(t_0) & \cdots & \phi_{nn}(t_0) \end{bmatrix}_{n\times n} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}_{n\times 1} = 0$$

For trivial solution $\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = 0$

$$\begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \phi_{21}(t) & \cdots & \phi_{n2}(t) \\ \vdots & & & \vdots \\ \phi_{n1}(t) & \phi_{n2}(t) & \cdots & \phi_{nn}(t) \end{bmatrix} \neq 0$$

EX: Homogenous system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Phi_1(t) = \begin{bmatrix} 3e^{2t} \\ -e^{2t} \end{bmatrix} \qquad \Phi_2(t) = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}$$

Solution? Linear independent?

Sol: choose t=0

$$\Phi_1(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad \Phi_2(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|\Phi_1(0), \Phi_2(0)| = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 4 \neq 0$$

$\therefore \Phi_1(t)$ & $\Phi_2(t)$ $\ldots\ldots \Phi_n(t)$ are linearly independent.

(4) General solution

If $\Phi_1(t)$ & $\Phi_2(t)$ $\ldots\ldots \Phi_n(t)$ are linearly independent

Solutions of X'=AX, then a linear combination of these solutions are

$C_1\Phi_1 + C_2\Phi_2 + ...C_n\Phi_n$ is the general solutions of X'=AX

EX:

$$\phi(t) = C_1\Phi_1 + C_2\Phi_2$$
$$= C_1\begin{bmatrix} 3e^{2t} \\ -e^{2t} \end{bmatrix} + C_2\begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix} \quad \text{is general solution of} \quad X' = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} X$$

(5)Matrix product of general solution

A. Fundamental matrix Ω:

A fundamental matrix for X'=AX is an nxn matrix where column are linearly independent solution of these system.

$$\Omega = \begin{bmatrix} \phi_1(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi_2(t) & \phi_{22}(t) & & \phi_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \phi_n(t) & \phi_{n2}(t) & & \phi_{nn}(t) \end{bmatrix}$$

B. General solution ΩC:

The general solution of X'=AX i.e. written as the matrix product of Ω

and C $\quad \Omega C = [\Phi_1 \quad \Phi_2 \quad \cdots \quad \Phi_n]\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$

Ex: Fundamental matrix for $X' = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} X$

i.e.

$$\Omega = \begin{bmatrix} 3e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix} \quad \therefore \text{ general solution}$$

$$\phi(t) = \Omega C = \begin{bmatrix} 3e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3C_1e^{2t} + C_2e^{6t} \\ -C_1e^{2t} + C_2e^{6t} \end{bmatrix}$$

3. General solution of nonhomogenous system X'=AX+G

Let Ω be a fundamental matrix for X'=AX and let ϕ_p be any solution of X'=AX+G. Then the general solution of X'=AX+G is $\Omega C + \phi_p$, where C is an nx1 matrix of arbitrary constants.

9.2 Solution of X'=AX when A is constant

1. Solution of X'=AX

Let A be an nxn matrix of number. Then $\xi e^{\lambda t}$ is a nontrivial solution of X'=AX iff λ is an eigenvalue of A with eigenvector ξ

Assume X=$\xi e^{\lambda t}$ is a solution of X'=AX

$$\left(\xi e^{\lambda t}\right)' = A\xi e^{\lambda t} \Rightarrow \xi\lambda e^{\lambda t} = A\xi e^{\lambda t}$$

$\Rightarrow A\xi = \lambda\xi$ eigenvalue problem

2. Linearly independent solutions

$\xi_1 e^{\lambda_1 t}, \xi_2 e^{\lambda_2 t}, \cdots \xi_n e^{\lambda_{n1} t}$ are n linearly independent solutions of X'=AX where $\xi_1, \xi_2, \cdots \xi_n$ are n linearly independent eigenvectors of nxn constant matrix a corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$

EX: $\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

Sol: A = $\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$ characteristic Eq.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -3 \\ +1 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 5) - 3$$

$$= \lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2) = 0$$

$\therefore$ eigenvalues $\lambda_1 = 2, \lambda_2 = 6$

Eigenvectors $\lambda_1 = 2$ $\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \xi_1 = 0$

$$-e_1 - 3e_2 = 0 \Rightarrow e_1 = -3, e_2 = 1$$

$$\xi_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\lambda_2 = 6$ $\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \xi_2 = 0$

$$-e_1 + e_2 = 0 \Rightarrow e_1 = 1, e_2 = 1$$

$$\xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\because \xi_1$ & ξ_2 linearly independent eigenvectors

$\therefore$ linearly independent solutions of the given systems are

$$\Phi_1(t) = \xi_1 e^{\lambda_1 t} = \begin{pmatrix} e^{2t} \\ -3e^{2t} \end{pmatrix}$$

$$\Phi_2(t) = \xi_2 e^{\lambda_2 t} = \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix}$$

$\therefore$ general solution is

$$\phi(t) = \Omega C = \begin{bmatrix} e^{2t} & e^{6t} \\ -3e^{2t} & e^{6t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

3. Solutions of $X' = AX$ for complex eigenvalues of A

(1) Solutions of complex value

$$A\xi = \lambda\xi$$

$$\therefore \overline{A\xi} = \overline{\lambda\xi}$$

if A is real matrix , then $A = \overline{A}$

$$\therefore A\overline{\xi} = \overline{\lambda}\overline{\xi}$$

$\therefore \lambda$ and $\overline{\lambda}$ are complex eigenvalues of A with eigenvalues ξ

and $\overline{\xi}$. Hence $\xi e^{\lambda t}$ and $\overline{\xi} e^{\overline{\lambda} t}$ are solutions of $x' = AX$

$\because \xi e^{\lambda t}$ and $\overline{\xi} e^{\overline{\lambda} t}$ are linearly independent($\because \lambda \neq \overline{\lambda}$)

$\therefore$ Fundamental matrix for complex eigenvalues is

$$\Omega = \left[\ \xi e^{\lambda t}, \overline{\xi} e^{\overline{\lambda} t}, \ldots.. \right]$$

(2) Real-valued solution

Let $\alpha + i\beta$ be an eigenvalue of an nxn real matrix A with corresponding eigenvector U+iV. Then $e^{\alpha t}[U\cos\beta t - V\sin\beta t]$ and $e^{\alpha t}[U\sin\beta t + V\cos\beta t]$ are linearly independent solutions of $x' = AX$

[proof]

$$\Phi_1(t) = (U + iV)e^{(\alpha+i\beta)t}$$

$$\Phi_2(t) = (U - iV)e^{(\alpha-i\beta)t}$$

From Euler formula, we have

$$\begin{cases} \Phi_1(t) = (U + iV)e^{\alpha t}(\cos\beta t + i\sin\beta t) \\ \Phi_2(t) = (U - iV)e^{\alpha t}(\cos\beta t - i\sin\beta t) \end{cases}$$

$$\therefore \frac{\Phi_1(t) + \Phi_2(t)}{2} = e^{\alpha t}[V\cos\beta t - V\sin\beta t]$$

$$\frac{\Phi_1(t) - \Phi_2(t)}{2} = e^{\alpha t}[V\cos\beta t + V\sin\beta t]$$

$\therefore e^{\alpha t}[U\cos\beta t - V\sin\beta t]$ and $e^{\alpha t}[U\sin\beta t + V\cos\beta t]$ are real-valued solutions of the original system and are linearly independent…

EX. Solve the system $\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

eigenvalues $\lambda_1 = 2, \lambda_2 = -1 + \sqrt{3}i, \lambda_3 = -1 - \sqrt{3}i$

eigenvalues $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 1 \\ -2\sqrt{3}i \\ -3 + \sqrt{3}i \end{pmatrix}, \xi_3 = \begin{pmatrix} 1 \\ 2\sqrt{3}i \\ -3 - \sqrt{3}i \end{pmatrix}$

solutions for $\lambda_1 = 2$ is $\Phi_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$

solutions for $\lambda = -1 \pm \sqrt{3}i \left(\alpha = -1, \beta = \sqrt{3}\right)$

eigenvector $\begin{pmatrix} 1 \\ 2\sqrt{3}i \quad i \\ -3 + \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2\sqrt{3} \\ \sqrt{3} \end{pmatrix} = U + iV$

$\therefore$ real-valued solutions for $\lambda = -1 \pm \sqrt{3}i$ are

$$\Phi_2(t) = e^{-t}[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cos\sqrt{3}t - \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \sin\sqrt{3}t]$$

$$\Phi_2(t) = e^{-t}[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cos\sqrt{3}t + \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \sin\sqrt{3}t] \therefore$$ general solution is

$\psi(t) = \Omega c$ where $\Omega = \lfloor \Phi_1, \Phi_2, \Phi_3 \rfloor$

4. Solution of X’=AX when A does not have n linearly independent eigenvectors

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix} = \begin{pmatrix} -2 & -1 & -5 \\ -25 & -7 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Eigenvalues of A are

$$|\lambda I - A| = (\lambda + 2)^3 = 0$$

$$\lambda 1 = \lambda 2 = \lambda 3 = -2$$

eigenvectors for $\lambda = -2$ B

$$E_1 = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}$$

One solution $\Phi_1(t) = E_1 e^{-et}$

2nd solution and 3rd solution refer to P395~P400 in text book

$$\Phi_2(t) = E_1 t e^{-2t} + E_2 e^{-2t}$$

$$\Phi_3(t) = \frac{1}{2} E_1 t^2 e^{-2t} + E_2 t e^{-2t} + E_3 e^{-2t}$$

5. Solution of X’= AX by diagonalizing A

X’= AX ………(1)

Let X=pz ………(2)

where matrix P is a matrix diagonalizing matrix A

∴X’= (pz)’= pz’ = AX…(3)

Premultiplying both sides of Eq.(3) by P^{-1}, we have

$P^{-1}Pz' = (P^{-1}AP)z$

Or

z’=DZ

Here $D = P^{-1}AP = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

Where λ1 λ2 ………λn are eigenvalues of A

$$\therefore \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix}' = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} \quad \text{uncoupled system}$$

$$\begin{aligned} Z_1' &= \lambda_1 Z_1 \rightarrow Z_1 = C_1 e^{\lambda_1 t} \\ Z_2' &= \lambda_2 Z_2 \rightarrow Z_2 = C_2 e^{\lambda_2 t} \\ &\vdots \\ Z_n' &= \lambda_n Z_n \rightarrow Z_n = C_n e^{\lambda_n t} \end{aligned}$$

∴general solution for Z’=DZ is

$$Z(t) = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \Omega_D(t) C$$

$\therefore X(t) = pz(t) = p\Omega_D(t)C$

9.3 Solution of X’ = AX + G

1. Variation of parameters

Nonhomogeneous system

X’ = AX + G…………….. (1)

Assume Ω(t) is a fundamental matrix for X’= AX, and

$\psi_p(t) = \Omega(t)u(t)$…………(2)

is a particular solution of X’= AX + G.

Substituting (2) into (1), we get

(Ωu)’ = A(Ωu) + G

Ω’u +Ωu’= A(Ωu) + G………………..(3)

Since ΩC is general solution of X’= AX, we have

(Ωc)’ = A(Ωc)

Ω’c = AΩc

Therefore

$\Omega' = A\Omega$ ……………(4)

Substutute (4) into (3) to get

$$A\Omega U + \Omega U' = A(\Omega U) + G$$

$$\Omega U' = G$$

Because Ω is nonsingular， Ω have an inverse

$$\Omega^{-1}\Omega U^{-1} = \Omega^{-1}G$$

$$U' = \Omega^{-1}G$$

$$\therefore U(t) = \int \Omega^{-1}G\,dt$$

The general solution of this nonhomogeneous system is

$$X(t) = \Omega(t)C + \Omega(t)u(t)$$

2. Solution of $X' = AX + G$ by diagonalizing A nonhomogeneous system

$$X' = AX + G \quad \cdot \quad \cdot \quad \cdot \ (1)$$

where A is a constant diagonalizable matrix

$$\therefore x = pz \quad \cdot \quad \cdot \quad \cdot \ (2)$$

Substituting (2) into (1), we have

$$(PZ)' = A(PZ) + G$$

$$PZ' = (AP)Z + G \quad \ldots.\ (3)$$

Premultiply both sides of (3) by p^{-1} to get

$$P^{-1}PZ' = P^{-1}(AP)Z + P^{-1}G$$

or

$$Z' = DZ + P^{-1}G$$

Here $D = P^{-1}AP = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ \vdots & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$

$$P^{-1}G = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\therefore \begin{cases} Z_1 = \lambda_1 Z_1 + f_1(t) \\ Z_2 = \lambda_2 Z_2 + f_2(t) \\ \vdots \\ Z_n = \lambda_n Z_n + f_n(t) \end{cases}$$

An uncoupled system of n independent DE's for Z.

$\therefore$ Solution of original system is

$$X = PZ$$

Here $P=[E_1, E_2, \ldots, E_n]$, $E_1, E_2, \ldots, E_n$ are eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.