CHAPTER 8

Eigenvalues, Diagonalization and Special Matrices

8.1 Eigenvalues and Eigenvectors

1. Definition:

A number λ is said to be an eigenvalue of matrix A if and only if there exists a nonzero $n \times 1$ matrix (vector) E satisfying the linear system.

$AE = \lambda E \quad \cdots (1)$

Where E is an eigenvector corresponding to the eigenvalue λ

Ex. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\therefore \begin{cases} \lambda = 0 ...eigenvalue..of.. \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ E = 0 ...eigenvector..of.. \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{cases}$$

2. Characteristic equation of eigenvalue

Alternative from of Equation (1)

$\lambda E - AE = (\lambda I - A)E = 0 \quad \cdots$ Homogeneous linear system

(n equation, n unknown)

For a nonzero solution E, we must have

$\left| \lambda I - A \right| = 0 \quad \cdots$ Characteristic equation of matrix A

Or

$$P_A(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0 = 0$$

$\cdots$ Characteristic polynomial of A

(1) Eigenvalues λ' s and the roots of the characteristic equation P (λ) = 0

(2) Corresponding eigenvectors Es' are the nonzero solution of $(\lambda I - A)E = 0$

Ex. Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

Sol: characteristic equation of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda-1)^2(\lambda+1) = 0$$

$\therefore \lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$

(1) $\lambda_1 = 1$

Eigenvector E satisfies $(I - A)E = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$e_2 = 0, e_3 = 0, e_1 = \alpha(\text{任意數}) \quad \therefore E = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$

(2) $\lambda_3 = -1$

Eigenvector E satisfies $(-I - A)E = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2e_1 + e_2 = 0 \Rightarrow e_2 = 2e_1$

$-2e_2 - e_3 = 0 \Rightarrow e_3 = -4e_1$

$e_1 = \beta$

$$\therefore E = \begin{pmatrix} \beta \\ 2\beta \\ -4\beta \end{pmatrix}$$

8.2 Diagonalization of Matrices

1. Diagonal matrix

(1)Definition

A matrix D = $\lfloor d_{ij} \rfloor_{n\times n}$ is a diagonal matrix if $d_{ij} = 0$ for $i \neq j$

Ex. $D = \begin{bmatrix} d_{11} & & & O \\ & d_{12} & & \\ & & \ddots & \\ O & & & d_{nn} \end{bmatrix}$

(2) Properties of diagonal matrix

$$D = \begin{bmatrix} d_1 & & & O \\ & d_2 & & \\ & & \ddots & \\ O & & & d_n \end{bmatrix} \quad W = \begin{bmatrix} w_1 & & & O \\ & w_2 & & \\ & & \ddots & \\ O & & & w_n \end{bmatrix}$$

(i) $DW = WD$

(ii) $|D| = d_1 d_2 \ldots d_n$

(iii) D is nonsingular iff $|0| \neq 0$

(iv) If $|D| \neq 0$, then $D^{-1} = \begin{bmatrix} 1/d_1 & & & O \\ & 1/d_2 & & \\ & & \ddots & \\ O & & & 1/d_n \end{bmatrix}$

(v) Eigenvalues of D are $\lambda_n = d_n$, $n = 1,2,\cdots,n$

2. Diagonalizable Matrix

(1) Definition

Matrix $A_{n\times n}$ is diagonalizable if and only if there exists a matrix $D_{n\times n}$ such that $P^{-1}AP = D$ is a diagonal matrix, matrix P diagonalizes A.

(2) Condition for Diagonalizability

If a $n\times n$ matrix A has n linearly independent eigenvectors, then A is diagonalizable.

Further, if P is $n\times n$ matrix having these eigenvectors of A as columns, there $P^{-1}AP$ is the diagonal matrix having the corresponding eigenvalues of A.

[Proof]

Let $V_1, V_2, \ldots\ldots, V_n$ be eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ and $P = (V_1, V_2, \ldots\ldots, V_n)$

First, prove that p is nonsingular $P = (V_1, V_2, \ldots\ldots, V_n)$ and $V_1, V_2, \ldots\ldots V_n$ are linearly independent.

➔Dimension of column space of matrix P=n

➔rank (P) =n

➔ $P_R = I_n$ (P_R: reduced matrix of P)

➔P is nonsingular

➔ P^{-1} exists

Then, we will prove that $P^{-1}AP = \begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_n \end{bmatrix}$

$AP = [AV_1, AV_2, ...AV_n]$

$\because AV_j = \lambda_j V_j \quad \therefore AP = [\lambda_1 V_1, \lambda_2 V_2, ...\lambda_n V_n]$

$P^{-1}AP = P^{-1}[\lambda_1 V_1, \lambda_2 V_2, ...\lambda_n V_n] = [\lambda_1 P^{-1}V_1, \lambda_1 P^{-1}V_1, ...\lambda_n P^{-1}V_n]$

$\because P^{-1}P = P^{-1}[V_1, V_2, ...V_n] = [P^{-1}V_1, P^{-1}V_2, ...P^{-1}V_n]$

$$= I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\therefore P^{-1}V_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} <== j^{th} - row$$

Hence $P^{-1}AP = \begin{bmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{bmatrix}$

(3) Criterion for Diagonalizability

A matrix $A_{n\times n}$ is diagonalizable iff A has n linearly independent eigenvectors. (充要條件)

If $Q^{-1}AQ$ is a diagonal matrix, there

1. Diagonal elements are eigenvalues of A, and
2. Columns of Q are eigenvectors of A

[Proof]

$$QD = \begin{pmatrix} |\ldots|\ldots\ldots| \\ V_1V_2\ldots V_n \\ |\ldots|\ldots\ldots| \end{pmatrix} D = \begin{pmatrix} |\ldots\ldots\ldots\ldots|\ldots\ldots\ldots| \\ d_1V_1.d_2V_2\ldots.d_nV_n \\ |\ldots\ldots\ldots\ldots|\ldots\ldots\ldots| \end{pmatrix}$$

$$AQ = A\begin{pmatrix} |\ldots|\ldots\ldots| \\ V_1V_2\ldots V_n \\ |\ldots|\ldots\ldots| \end{pmatrix} = \begin{pmatrix} |\ldots\ldots\ldots|\ldots\ldots\ldots| \\ AV_1.AV_2\ldots.AV_n \\ |\ldots\ldots\ldots|\ldots\ldots\ldots| \end{pmatrix}$$

Since AQ=QD, then column j of AQ equals column j of QD, so

$AV_j = d_jV_j$ which proves that d_j is an eigenvalue of A with associated

eigenvector V_j.

(4) Sufficient condition for Diagonalizability

If matrix A_{n*n} has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent, and A is diagonalizable.

Ex. Diagonalize $A = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}$ if possible?

Sol. Find eigenvalues & eigenvector of A

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 4 \\ 0 & \lambda-3 \end{vmatrix} = (\lambda+1)(\lambda-3) = 0$$

$\lambda_1 = -1 \neq \lambda_2 = 3$

associated eigenvectors are $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ linearly independent

$\Rightarrow$ A is diagonalizable

$\therefore$ Matrix P diagonalizing A is $P = [V_1, V_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Thus $P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Note: (i) $Q = [3V_1 - 2V_2] = \begin{bmatrix} 3 & -2 \\ 0 & -2 \end{bmatrix}$ also diagonalize A

$$Q^{-1}AQ = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

(ii) $S = [V_2, V_1] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ also diagonalize A

$$S^{-1}AS = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

Ex. $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ Diagonalizable?

Sol. $|\lambda I - B| = \begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$

$\lambda_1 = \lambda_2 = 1$ repeated roots.

From the system $(I - B)X = 0$ $\quad X = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ for all

Any two eigenvectors are linearly dependent

$\Rightarrow$ A is not diagonalizable.

8.3 Orthogonal and Symmetric Matrices

1. Orthogonal matrix

(1) Definition

A nonsingular matrix $A_{n\times n}$ is orthogonal matrix if and only if

$A^t A = AA^t = I$ or $A^{-1} = A^t$

(2) Theorem

Matrix $A_{n\times n}$ is orthogonal if and only if A^t is orthogonal.

pf. $(A^t)^t A^t = I = A^t (A^t)^t$ $\qquad [(A^t)^t = A]$

$\Rightarrow$ A is orthogonal

(3) Theorem

$A_{n\times n}$ is orthogonal $\quad |A| = \pm 1$

pf. $|AA^t| = |I| = 1$

$|AA^t| = |A||A^t| = |A||A| = |A|^2$

$\Rightarrow |A| = \pm 1$

(4) Orthonormality of row (column) vectors

A real matrix $A_{n\times n}$ is orthogonal iff the row (column) vectors are orthonormal in R^n .

orthonormal vectors:

vectors and orthogonal and each has norm I.

pf. $(AA^t)_{ij} = \sum_{s=1}^{n} a_{is} a^t_{sj} = \sum_{s=1}^{n} a_{is} a_{js}$ $\quad (a^t_{sj} = a_{js})$

Assume rows of A and orthonormal vectors

$$\therefore \sum_{s=1}^{n} a_{is} a_{js} = \begin{cases} 0 & i \pm j \\ 1 & i = j \end{cases}$$

$\because AA^t = I_n \Rightarrow$ A is orthogonal

Ex: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ orthogonal?

Sol: $AA^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

2. Symmetric Matrices

(1) Defination

Matrix $A_{n\times n}$ B symmetric if and only if $A = A^t$, namely $a_{ij} = a_{ji}$

Ex:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 4 \\ 0 & 4 & 2 \end{pmatrix} = A^t$$

(2) Real eigenvalues

Theorem

If matrix A is real symmetric matrix then the eigenvalues of A are real.

pf. E is eigenvector associated with eigenvalue λ of A.

∴ AE=Aλ conjugate of last eqn. is , $\overline{A}\overline{E} = \overline{\lambda}\overline{E}$ or $A\overline{E} = \overline{\lambda}\overline{E}$

($\because A = \overline{A}$) real

$$(A\overline{E})^t = \overline{E}^t A^t = \overline{E}^t A = (\overline{\lambda}\overline{E})^t = \overline{E}^t \overline{\lambda}^t = \overline{\lambda}\overline{E}^t$$

$$\because \overline{E}^t A = \overline{\lambda}\overline{E}^t$$

Post multiplying above eqn. with E , we have

$$\overline{E}^t AE = \overline{\lambda}\overline{E}^t E \cdots\cdots 1$$

Premultiplying AE=λE with $\overline{E}^t$, we have

$$\overline{E}^t AE = \lambda \overline{E}^t E \cdots\cdots 2$$

2-1 yields

$$(\lambda - \overline{\lambda})\overline{E}^t E = 0$$

$$\because \overline{E}^t E = |e_1|^2 + |e_2|^2 + \cdots\cdots + |e_n|^2 > 0 (\because E \neq 0)$$

$$\therefore \lambda - \overline{\lambda} = 0$$

∴λ is a real number

(3) Orthogornal eigenvectors

Theorem

If A is real symmetric matrix , then eigenvectors corresponding to different eigenvalues are orthogonal.

pf. eigenvaluesλ&μ, eigenvalues E&G

∴AE=λE , AG=μG , Now prove $E \cdot G = E^t G = 0$

$$\lambda E^t G = (\lambda E)^t G = (AE)^t G = E^t A^t G = E^t AG = E^t \mu G = \mu E^t G$$

$\therefore (\lambda - \mu) E^t G = 0$ if $\lambda \neq \mu$ Then $E^t G = 0 \Rightarrow E \,\&\, G$ orthogonal

(4) Orthogonal Diagonalization

Theorem

If matrix A is real symmetric matrix , then there is a real orthogonal matrix that diagonalizes A.

$$P^{-1}AP = D = P^t AP \quad (P^{-1} = P \quad orthogonal \quad matrix \quad)$$

Here,

$$P = \left[\frac{E_1}{\sqrt{E_1^t E_1}}, \frac{E_2}{\sqrt{E_2^t E_2}}, \cdots\cdots, \frac{E_n}{\sqrt{E_n^t E_n}} \right]$$

Where $E_1, E_2, \cdots\cdots, E_n$ are linearly independent eigenvectors of A

corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Ex: $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ real symmetric matrix

Sol: $|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 0 & -2 \\ 0 & \lambda - 2 & 0 \\ -2 & 0 & \lambda \end{vmatrix} = 0$

eigenvalues $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 4$

eigenvectors E_1, E_2, E_3

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} E_1 = 0 \qquad , E_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & 0 \end{bmatrix} E_2 = 0 \qquad , E_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} E_3 = 0 \qquad , E_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

E_1, E_2, E_3 are orthogonal to one another.

$$E_1^t E_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$$

$$E_1^t E_3 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$E_2^t E_3 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = 0$$

Consider vectors V_1, V_2, V_3 as

$$V_1 = \frac{E_1}{|E_1|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$V_2 = \frac{E_2}{|E_2|} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$V_3 = \frac{E_3}{|E_3|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

Then V_1, V_2, V_3 are orthogonal

Here $P=(V_1 \quad V_2 \quad V_3)=\begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$ is an orthogonal matrix

$\because V_1, V_2, V_3$ are linearly independent eigenvectors of A

$\Rightarrow$ matrix P diagonalizes A.

$$\therefore P^{-1}AP = P^tAP = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

8.4 Quadratic Forms

1. Complex quadratic form

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\bar{z}_j z_k$$

a_{jk} : complex number

z_j : complex variable

(1) square terms

$$\bar{z}_k z_j = \bar{z}_j z_j = |z_j|^2 \quad , \quad \text{j=k}$$

(2) mixed product terms

$$\bar{z}_k z_j \quad , \quad j \neq k$$

(3) matrix form

$\overline{Z}^t AZ$, A= $\lfloor a_{ij} \rfloor$

2. Real Quadratic form

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

a_{jk} : real number

x: real variable

(1) square terms

x_j^2 , j=k

(2) mixed product terms

$x_j x_k$, $j \neq k$

(3) matrix form

$X^t AX$

$$=(X_1, X_2, \cdots\cdots X_n)\begin{bmatrix} a_{11}a_{12}\cdots\cdots a_{1n} \\ a_{21} \\ \vdots \\ a_{n1}\cdots\cdots\cdots a_{nn} \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

Ex. Given: $2X_1^2 - 7X_1X_2 + 9X_2^2$

Find: matrix form

$$2X_1^2 - 7X_1X_2 + 9X_2^2$$

$$=(X_1 X_2)\begin{bmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$$

$$\therefore \begin{cases} a_{11} = 2 \\ a_{12} + a_{21} = -7 \\ a_{22} = 9 \end{cases}$$

$$\therefore A = \begin{bmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{bmatrix} \qquad \text{if} \quad a_{12} = a_{21} = -\frac{7}{2}$$

$$=\begin{bmatrix} 2 & 2 \\ -9 & 9 \end{bmatrix} \qquad \text{then} \quad A=\begin{bmatrix} 2 & -\frac{7}{2} \\ -\frac{7}{2} & 9 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -9 \\ 2 & 9 \end{bmatrix} \qquad \text{Symmetric matrix}$$

$$=\begin{bmatrix} 2 & -3 \\ -4 & 9 \end{bmatrix}$$

3. problems mvolving quadratic form

(1) Eigenvalue of matrix

$$\lambda = \frac{\overline{E}^t AE}{\overline{E}^t E}$$

(2) kinetic energy of a system of particles

$$T = \sum \frac{1}{2} m v_i^2$$

(3) Equation of a conic

$$ax^2 + bxy + cy^2 = d$$

4. principal Axis theorem

The transformation of coordinates $X = QY$

transforms the quadratic form $\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$

to the "stardard form" $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots\cdots + \lambda_n y_n^2 = \sum_{j=1}^{n} \lambda_j y_j^2$

Here Q is an orthogonal matrix that diagonalizes the real symmetric matrix $A_{n\times n}$, and $\lambda_1, \lambda_2, \cdots\cdots \lambda_n$ are eigenvalue of A

[p.f.]

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

$$= X^t AX$$

$$=(QY)^t A(QY)$$

$$=Y^t (Q^t AQ)Y$$

$$=\bar{Y}^t\left(Q^{-1}AQ\right)Y$$

$$=\left(y_1, y_2 \cdots\cdots y_n\right)\begin{bmatrix}\lambda_1 & & 0\\ \vdots & & \vdots\\ \vdots & \lambda_2 & \vdots\\ 0 & & \lambda_n\end{bmatrix}\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix}$$

$$=\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2+\cdots\cdots+\lambda_n y_n^2$$

$$=\sum_{j=1}^{n}\lambda_j y_j^2$$

Ex. Guadratic form $x_1^2-2x_1x_2+x_2^2$

Sol: In matrix form X^tAX

$$x_1^2-2x_1x_2+x_2^2$$

$$=\left(x_1x_2\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$\therefore A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ real symm matrix

Eigenvalues & eigenvectors of A

$$\lambda_1=0 \qquad E_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\lambda_2=2 \qquad E_2=\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

Orthogonal matrix Q diagonalizing A is

$$\text{Q}=\left[E_1 \quad E_2\right]=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$\therefore$ coordinate transformation is

$$X=QY$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\therefore$ standard form of $x_1^2 - 2x_1x_2 + x_2^2$ is

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 2y_2^2$$

If $x_1^2 - 2x_1x_2 + x_2^2 = 4$

$$(x_1 - x_2)^2 = 4$$

$$x_1 - x_2 = \pm 2$$

then after X= $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ Y transformation,

$2y_2^2 = 4$ or $y_2 = \pm\sqrt{2}$

Q= $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$

8.5. Vniting, Hermition and
skow Hermition matrices

1. Complex matrix

A= $\begin{bmatrix} a_{11}a_{12} \cdots\cdots a_{1n} \\ a_{21}a_{22} \cdots\cdots a_{2n} \\ \vdots \\ a_{n1} \cdots\cdots\cdots a_{nn} \end{bmatrix}$

a_{ij} = complex number

Dot product of two complex vector in C^n

$$z \cdot w = \bar{z}_1 w_1 + \cdots\cdots\cdots + \bar{z}_n w_n$$

$$=\sum_{k=1}^{n}\bar{z}_k w_k$$

In matrix form

$$(\bar{z}_1,\bar{z}_2,\cdots\cdots\bar{z}_n)\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}=\bar{z}^t w=\sum_{k=1}^{n}\bar{z}_k w_k$$

2. Unitary matrix

(1) Definition

A complex matrix $U_{n\times n}$ is unitary if and only if

$$\bar{U}^{-1}=U^t \quad \text{or} \quad \bar{U}U^t=I_n$$

If U is real matrix

then $\bar{U}=U$

$$\therefore \bar{U}U^t=UU^t=I_n$$

$\Rightarrow$ Unitary matrix is orthogonal

$\therefore$ Unitary matrix is a complex analogue of orthogonal matrix

Ex. $U=\begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\bar{U}=\begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \bar{U}U^t=\begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I_2$$

$\therefore$ U is unitary matrix

(2)Unitary system of vectors complex n-vectors $\vec{F}_1, \vec{F}_1 ... \vec{F}_1$ from a unitary system if $\vec{F}_i \cdot \vec{F}_j = \begin{Bmatrix} 0 & i \pm j \\ 1 & i = j \end{Bmatrix}$ complex analogue of orthonormaket

(3)Condition for unitary matrix A complex matrix $U_{n\times n}$ is unitary if and only if it's row (column) vectors from a unitary system

Ex:

$$U=\begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$ Unitary matrix=? Unitary system=?

Sol:

$$\overline{U}U^t=\begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$\therefore$ U is a unitary matrix

row vectors=$\overrightarrow{U_1} = \left(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\overrightarrow{U_2} = \left(-\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\overrightarrow{u_1} \cdot \overrightarrow{u_2} = \overline{(\frac{i}{\sqrt{2}} \frac{1}{\sqrt{2}})} \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \left(-\frac{i}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 0$$

$$\overrightarrow{u_1} \cdot \overrightarrow{u_1} = -\left|\frac{i}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1 = \overrightarrow{u_2} \cdot \overrightarrow{u_2}$$

3. Hermitian matrix

A complex matrix $H_{n\times n}$ is Hermitian if and only if $\overline{H} = H^t$.

If H is real, then $\overline{H} = H = H^t$, and H is symmetric.

4. skew- Hermitian matrix

A complex matrix $S_{n\times n}$ is skew- Hermitian if $\overline{S} = S^t$.

If S is real, then $\overline{S} = S = S^t$, and S is skew-symmetric.