Chapter 7

Determinants

7-1 Permutations

1. Permutation

   A permutation p of order n is an arrangement of the integers 1,2…n is any order.

Ex. n=2 permutations (2!)

1,2 p (1)=1,p (2)=2

2,1 p (1)=2,p (2)=1

Ex. n=3 permutations (3!)

1,2,3

1,3,2

2,1,3

2,3,1

3,1,2

3,2,1 p (j)=the number the permutation has put in place j.

2. Even and Odds permutations (奇排列和偶排列)

   Rules:

   (1) For each number k in the permutation, count the number of integer to its right that are smaller than k.
   (2) Sum all the numbers obtained in (1) in order to get a number.
   (3) If the number in (2) is odd, then the permutation is called odd permutation, otherwise even permutation.

Ex. permutation 2,5,1,4,3

k	$number\_of\_integers < k$
2	1
5	3
1	0
4	1
3	0
Sum	5

$\therefore$ 2,5,1,4,3 is an odd permutation!

Ex. Permutation 2,1,5,4,3

k	$numbers\_of\_intergers < k$
2	1
1	0
5	2
4	1
3	0
Sum	4

$\therefore$ 2,1,5,4,3 is an even permutation!

3. Sign of permutation

$$\text{sgn(p)}=\begin{cases} 0 & if \quad p \quad is \quad even \\ 1 & if \quad p \quad is \quad odd \end{cases}$$

7.2 Definition of determinate

1. Definition

$$\det(A)=\sum_p (-1)^{\text{sgn}(p)} a_{1p(1)}a_{2p(2)}\cdots a_{np(n)}$$

Where p (j) is determined by the permutations on 1, 2…n

2. Example

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(A)=\sum_p (-1)^{\text{sgn}(p)} a_{1p(1)}a_{2p(2)}$$

n=2

permutations	even or odd	P(1)	P(2)	sgn(p)
1,2	Even	1	2	0
2,1	Odd	2	1	1

$$\therefore \det(A)=(-1)^0 a_{11}a_{22}+(-1)^1 a_{12}a_{21}$$
$$=a_{11}a_{22}-a_{12}a_{21}$$

7.3 Properties of determinate

1.Theorem

A_{nxn}, B_{nxn}

$|A|=0$ if A has a zero row

2. Theorem (Type Ⅱ row operation)

$|B| = \alpha |A|$ if B is obtained from, A by multiplying row k by a scalar α

3.Theorem (Type Ⅰ row operation)

$|A| = -|B|$ if B is formed from A by interchanging two rows.

4. Corollary

$|A| = 0$ if A has two identical rows.

Pf: Assume that B is obtained from A by interchanging the two identical rows.

$\therefore B = A \Rightarrow |B| = |A| \ldots (1)$

But from theorem3, we have

$|B| = -|A| \ldots (2)$

$\therefore (1) + (2)$

$|B| = |A| = 0$

5.Corollary

$|A| = 0$ if row k of A is α times row i.

P.f:

case (1) α= 0

$|A| = 0$ (Theorem Ⅰ)

case (2) α≠0

If B is obtained from A by multiplying know by 1/α,

then B has two identical row i & k

$\therefore |B| = 0$ (corollary 4) ...... (1)

But from theorem 2, we have

$|B| = \frac{1}{\alpha} |A| \ldots\ldots (2)$

∴From (1) & (2), we have

$|A| = 0$

6.Theorem

$|AB| = |A||B|$

7.Theorem

$$|A| = |A_1| + |A_2|$$

if $$|A| = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{ij} & \cdots & \alpha_{1n} \\ \vdots & & & & \vdots \\ \alpha_{k1}+\beta_{kj} & \cdots & \alpha_{kj}+\beta_{kj} & \cdots & \alpha_{kn}+\beta_{kn} \\ \vdots & & & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nj} & \cdots & \alpha_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1n} \\ \vdots & & & \vdots \\ \alpha_{k1} & \cdots & \cdots & \alpha_{kn} \\ \vdots & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nn} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1n} \\ \vdots & & & \vdots \\ \beta_{k1} & \cdots & \cdots & \beta_{kn} \\ \vdots & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nn} \end{vmatrix}$$

$$= |A_1| + |A_2|$$

8. Theorem (Type Ⅲ row operation)

$|B| = |A|$ If B is formed by A by adding r times row I to row k

P.f.

$$|B| = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{ij} & \cdots & \alpha_{1n} \\ \vdots & & & & \vdots \\ \alpha_{i1} & \cdots & \cdots & \cdots & \alpha_{in} \\ \alpha_{i1}+\alpha_{k1} & \cdots & \cdots & \cdots & \alpha_{in}+\alpha_{kn} \\ \vdots & & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \cdots & \alpha_{nn} \end{vmatrix}$$

$$\underline{\underline{\text{Theorem7}}} \begin{vmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1n} \\ \vdots & & & \vdots \\ \alpha_{i1} & & & \alpha_{in} \\ \vdots & & & \vdots \\ \alpha_{i1} & & & \alpha_{in} \\ \vdots & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nn} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1n} \\ \vdots & & & \vdots \\ \alpha_{i1} & & & \vdots \\ \vdots & & & \vdots \\ \alpha_{k1} & & & \vdots \\ \vdots & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nn} \end{vmatrix}$$

$$\underline{\underline{\text{Theorem}}}\, r \begin{vmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1n} \\ \vdots & & & \vdots \\ \alpha_{i1} & & & \alpha_{in} \\ \vdots & & & \vdots \\ \alpha_{i1} & & & \alpha_{in} \\ \vdots & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nn} \end{vmatrix} + |A|$$

$$\underline{\underline{\text{Corollany4}}} \qquad 0 + |A|$$

$$\therefore |B| = |A|$$

9. Theorem

$$|A| = |A^T|$$

7.4 Evaluation of Determinants by Elementary row operations

1. Method

Given an n×n matrix A, us the row operations to obtain a new matrix B having at most one nonzero element is some row I or column j

Then $|A|$ is a scalar multiple of $|B|$ is a scalar multiple of the (n-1) × (n-1) determinant formed by deleting from B the row and column containing this nonzero element.

$$\begin{vmatrix} \alpha_{11} & \cdots & \alpha_{ij} & \cdots & \alpha_{1n} \\ \vdots & & & & \vdots \\ \alpha_{i-11} & & & & \alpha_{i-1n} \\ 0 & 0 & \alpha_{ij} & 0 & 0 \\ \alpha_{i+1} & & & & \vdots \\ \vdots & & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \cdots & \alpha_{nn} \end{vmatrix}_{n\times n}$$

$$= (-1)^{i+j} \alpha_{ij} \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1j-1} & \alpha_{1j+1} & \cdots & \alpha_{1n} \\ \vdots & & & & & \vdots \\ \alpha_{i-11} & \cdots & \cdots & \cdots & \cdots & \alpha_{i-1n} \\ \alpha_{i+11} & \cdots & \cdots & \cdots & \cdots & \alpha_{i+1n} \\ \vdots & & & & & \\ \alpha_{n1} & \cdots & \cdots & \cdots & \cdots & \alpha_{nn} \end{vmatrix}$$

2. Example

Given $A: \begin{pmatrix} -2 & 6 & 5 & -3 \\ 4 & 4 & -8 & 2 \\ 6 & 3 & 3 & -6 \\ 8 & 9 & -11 & 4 \end{pmatrix}$

Find: $|A|$ by row operations.

Sol:

$$A = \begin{pmatrix} -2 & 6 & 5 & -3 \\ 4 & 4 & -8 & 2 \\ 6 & 3 & 3 & -6 \\ 8 & 9 & -11 & 4 \end{pmatrix} \times(-1/2)$$

$$\Rightarrow B = \begin{pmatrix} 1 & -3 & -5/2 & -3/2 \\ 4 & 4 & -8 & 2 \\ 6 & 3 & 3 & -6 \\ 8 & 9 & -11 & 4 \end{pmatrix} \begin{matrix} \downarrow \\ \lrcorner \times(-4) \\ \lrcorner \times(-6) \\ \lrcorner \times(-8) \end{matrix}$$

$$\Rightarrow C = \begin{pmatrix} 1 & -3 & -5/2 & -3/2 \\ 0 & 16 & 2 & -4 \\ 0 & 21 & 18 & -15 \\ 0 & 33 & 9 & -8 \end{pmatrix}$$

$$\therefore |B| = -\frac{1}{2}|A| \quad \text{(Theorem 2)}$$

$$|C| = |B| \quad \text{(Theorem 8)}$$

$$D = \begin{pmatrix} 16 & 2 & -4 \\ 21 & 18 & -15 \\ 33 & 9 & -8 \end{pmatrix}$$

Formed form C by deleting row1 and column 1

$$\therefore |D| = \frac{1}{(-1)^{1+1} C_{11}} |C| = |C|$$

$$D = \begin{pmatrix} 16 & 2 & -4 \\ 21 & 18 & -15 \\ 33 & 9 & -8 \end{pmatrix} \xrightarrow{\times(-\frac{1}{2})} E = \begin{pmatrix} 8 & 1 & -2 \\ 21 & 18 & -15 \\ 33 & 9 & -8 \end{pmatrix} \begin{matrix} \downarrow \\ \lrcorner \times(-18) \\ \lrcorner \times(-9) \end{matrix}$$

$$\longrightarrow F = \begin{pmatrix} 8 & 1 & -2 \\ -123 & 0 & 21 \\ -39 & 0 & 10 \end{pmatrix}$$

$$\therefore |E| = \frac{1}{2}|D|, \quad |F| = |E|$$

$$|F| = (-1)^{1+2}\begin{vmatrix} -123 & 21 \\ -39 & 10 \end{vmatrix} = 411$$

$$\therefore |A| = -2|B| = -2|C| = -2|D| = -4|E| = -4|F|$$

$$= -4 \times 411$$
$$= -1644$$

7.5 Cofactor Expansions

1. Minor, M_{ij}

   The minor of element a_{ij} of matrix A is the determinant of the (n-1) × (n-1) matrix obtained by deleting row i and column j of A.

   Ex.

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

   Sol: minor of a_{12}=M_{12}=$\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix}$=-6

   minor of a_{33}=M_{33}=$\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}$=-3

2. Cofactor, C_{ij}

   The cofactor of a_{ij} is the number $(-1)^{i+j}M_{ij}$

   Ex.

   Cofactor of a_{12} = c_{12} = $(-1)^{1+2}M_{12}$= 6

   Cofactor of a_{33} = c_{12} = $(-1)^{3+3}M_{33}$= -3

3. Cofactor expansion by a row

   If A is n×n, then for any integer I with $1 \le i \le n$, $|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$

   P.f.

$$|A| = \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$$
= \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ a_{i1} & 0 & \cdots & \cdots & 0 \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ 0 & a_{i2} & \cdots & \cdots & 0 \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix} + \cdots\cdots
$$

$$
+ \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix}
$$

$$
= (-1)^{i+1} a_{i1} M_{i1} + (-1)^{i+2} a_{i2} M_{i2} + \cdots\cdots + (-1)^{i+n} a_{i+n} M_{in}
$$

$$
= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{(i dummy index)}
$$

Ex.

$$
A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}
$$

Determine $|A|$ by cofactor expansion by a row.

Sol: Cofactor expansion by row 1 (i = 1)

$$
\begin{aligned}
\therefore |A| &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} M_{1j} \\
&= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \\
&= \begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} + 7 \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}
\end{aligned}
$$

4. Cofactor expansion by a column

Let A be n×n. Then for any integer j with $1 \le j \le n$, $|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$

Ex.

$$
A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}, \quad |A| = ?
$$

Sol: Cofactor expansion by column 1, j=1

$$|A| = \sum_{i=1}^{3} (-1)^{i+j} a_{i1} M_{i1}$$

$$= (-1)^{1+1} a_{11} M_{11} + (-1)^{2+1} a_{21} M_{21} + (-1)^{3+1} a_{31} M_{31}$$

$$= \begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 7 \\ 5 & 8 \end{vmatrix}$$

7.6 Determinants of Triangular Matrices

1. Upper triangular matrix

   An n×n matrix A is called upper triangular if all the elements below the main diagonal are zero.

$$A: \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

2. Lower triangular matrix

   Matrix A is called lower triangular matrix if all element upper the main diagonal are zero.

3. Determinants of triangular matrices

$|A| =$ The product of its main diagonal element

$$= a_{11} \quad a_{22} \quad \cdots \quad a_{nn}$$

7.7 A Determinant formula for a matrix Inverse.

1. Condition for no singularity let A be n×n, then A is no singular iff $|A| \neq 0$

p.f. Assume $|A| \neq 0$

$\therefore |A| = \alpha |A_R| \neq 0$

$\therefore$ A_R can not have any zero rows

$\therefore$ $A_R = I_n$

$\Rightarrow$ Rank (A) = Rank (A_R) =n

$\Rightarrow$ A is no singular

Assume A is no singular

$\Rightarrow$ $A_R = I_n$

And: $|A| = \alpha |A_R|$

$|A_R| = |I_n| = 1$

$\therefore \ |A| = \alpha \neq 0$ (充分必要條件)

2. Formula for matrix Inverse let A be n×n no singular matrix.
   Then $B=A^{-1}$
   Where the element b_{ij} of B is given as follows $b_{ij} = \frac{1}{|A|}(-1)^{i+j} M_{ij}$

   Ex. $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

   $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2 \neq 0$ No singular

   $$\therefore A^{-1} = B = \left[\frac{1}{|A|}(-1)^{i+j} M_{ij} \right]$$

   $$= \frac{1}{|A|} \begin{bmatrix} (-1)^{1+1} M_{11} & (-1)^{1+2} M_{21} \\ (-1)^{2+1} M_{12} & (-1)^{2+2} M_{22} \end{bmatrix}$$

7.8 Cramer's Rule

Let A be nonsingular n×n matrix. Then the ungula solution of AX=B is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Where $x_k = \frac{1}{|A|}|A(K;B)|$ and $A(K;B)$ is the matrix obtained from A by replacing column K of A by matrix B.

Ex. $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

A=$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ no singular

A unique solution exists

$$x_1 = \frac{A(1;B)}{|A|} = \frac{\begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}}$$

$$x_2 = \frac{A(2;B)}{|A|} = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}}$$