CHAPTER 6

Matrices and systems of Linear Equations

6-1 Matrixes

1. Matrix

A matrix is any rectangular array of objects arranged in rows and in columns.

$$A=[\, a_{ij} \,] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Here aij is the entry or element of matrix A in row i and column j.

n x m	Matrix	example
n≠m	Rectangular	$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
n = m	Square	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
n =1	Row	$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times 4}$
m =1	Column	$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times 1}$

2. Equality of matrix

$$A = \lfloor a_{ij} \rfloor_{m\times n} = B = \lfloor b_{ij} \rfloor_{p\times q}$$

iff $m = p, n = q$ and $a_{ij} = b_{ij}$

3. Matrix algebra

(1) matrix addition

$$A = \lfloor a_{ij} \rfloor_{m\times n}, B = \lfloor b_{ij} \rfloor_{n\times m}$$

$$A + B = \lfloor a_{ij} + b_{ij} \rfloor_{m\times n}$$

(2) product of matrix and a scalar α

$$\alpha A = \lfloor \alpha a_{ij} \rfloor_{n\times m}$$

(3) multiplication of matrices

$$A_{n\times r} B_{r\times m} = \left[\sum_{s=1}^{r} a_{ij} b_{ij} \right]$$
$$= C_{n\times m}$$
$$= \left[C_{ij} \right]_{n\times m}$$

$$C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$
$$= (row \quad i \quad of \quad A) \bullet (column \quad j \quad of \quad B)$$

$$\begin{cases} AA = A^2 \\ AAA = A^3 \\ AA\cdots = A^n \end{cases} \qquad A^{100} = ? \quad if \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(4) Theorems

i. Commutative law of addition

A+B=B+A

ii. Associative law of addition

A+ (B+C) = (A+B) +C

iii. Distributive law

$A \cdot (B + C) = AB + AC$

iv. Distributive law

$(A + B) \cdot C = AC + BC$

v. Associative law of multiplication

(AB)C=A (BC)

(5) Other properties

i. $AB \neq BA$

ii. $AB = AC$ and $B \neq C$

Ex.

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 4 & 12 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 21 & 54 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 5 & 11 \end{bmatrix}$$

but $\begin{bmatrix} 4 & 12 \\ 3 & 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 7 \\ 5 & 11 \end{bmatrix}$

iii. $AB = 0$ and $A \neq 0, B \neq 0$

(6) Special Matrices

(i) zero matrix

$0 = \lfloor a_{ij} \rfloor_{n\times m}$ where $a_{ij} = 0$

(a) $A+0=0+A=0$

(b) $A+(-A)=0$

(ii) Identity matrix, I_n

$$I_n = \lfloor I_{ij} \rfloor_{n\times n} \quad \text{Square matrix} \Leftrightarrow I_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\boxed{I_n A_{n\times m} = A_{n\times m} I_m = A_{n\times m}}$$

(iii) Transpose(轉置)of matrix

$$A = \lfloor a_{ij} \rfloor_{n\times m} \qquad A^t = [a_{ij}]_{m\times n}$$

ij element of $A^t = ji$ element of A

(a) $I_n^t = I_n$

(b) $(A^t)^t = A$

(c) $(AB)^t = B^t A^t$

4. Matrices and systems of linear equations

(1) System of linear algebra equation

$$\begin{cases} 2x_1 + x_2 = 1 \\ 3x_1 + 2x_2 = 2 \end{cases}$$

In matrix form

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad Ax = B$$

where A=matrix of coefficients of system

X=matrix of unknown

B=matrix of constants

(2) system of linear differential eqn.

$$\begin{cases} x_1' + tx_2' - x_3' = f(t) \\ t^2 x_1' - \cos(t) x_2' - x_3' = g(t) \end{cases}$$

In matrix form

$$\begin{bmatrix} 1 & t-1 \\ t^2 & -\cos(t)-1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

$$AX' = F$$

Ex. System AX=0 $\begin{bmatrix} -1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \\ -3 & 1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$=0, the solution space=?

Sol. $A_R = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{9}{8} \\ 0 & 1 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 1 & \frac{-1}{4} \end{bmatrix}$

$\because m = 5$ rank(A)=rank(A_R)=4

$\therefore$ Dimension of solution space of AX=0

m-rank(A)=5-4=1

General solution is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \frac{-9}{8} \\ \frac{5}{8} \\ \frac{9}{8} \\ \frac{-1}{4} \\ 1 \end{bmatrix}$$ One arbitrary constant.

3. Trivial solution

The solution $X_1 = X_2 = \ldots\ldots = X_n = 0$ is the trivial solution of a homogeneous system AX=0.

4. Existence of nontrivial solution

(1)Theorem

Let A be n$\times$m matrix. The system AX=0 has a nontrivial solution iff m>rank(A).

$\because$ Dimension of solution space of AX=0=m-rank(A)>0

Then system AX=0 has a nonzero solution!

(2)Corollary

A. AX=0 always has a nontrivial solution if the number of unknowns exceed the number of equations.

A. $A_{n\times m}X_{m\times 1}=0$, n is the number of equations; m is the number of unknowns.

$\because$ n<m and rank(A) $\leq$ n. $\therefore$ m-rank(A) $\geq$ m-n>0.

B. A is an n×m matrix of real number. Then the system AX=0 has only the trivial solution iff $\boxed{A_R=I_n}$

$\because A_R=I_n$

$\therefore$ rank(A)=rank(A_R)=n, and m=n.

$\therefore$ m-rank(A)n-n=0

$\Rightarrow$ System AX=0 has only zero solution.

6-7 Nonhomogeneous Systems of Linear Equations

1. Nonhomogeneous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2m}x_m = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nm}x_m = b_n \end{cases}$$

I_n matrix from AX=B

When A=[a_{ij}]=n×m coefficient matrix

X=[X_j]=m×1 unknowns

B=[b_i]=n×1 matrix (at least one of which is nonzero).

Augmented matrix of system AX=B

$$[A \vdots B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1m} & b_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nm} & b_n \end{bmatrix}$$

2.Structure of solution of AX=B

(1)Theorem

Let Up be a solution of AX=B, then every solution of AX=Bi the form of U_p+H. in which H is a solution of AX=0.

[Proof]

Assume $\boxed{AW=B}$ $\because$ $\boxed{AU_p=B}$

$\therefore$ AW=AU$_p$=A(W-U$_P$)=B-B=0

$\therefore$ W-U$_P$=H is the solution of AX=0

$\therefore$ W=U$_P$+H

(2) Remark on U$_P$ and H

A. U$_P$=particular solution of AX=0

B. H=general solution of AX=0

C. U$_P$+H=general solution of AX=0

3. Existence and Uniquences of solutions

(1) Consistent system of equation

Definition:

AX=B is said to be consistent if there exists a solution, otherwise the system is inconsistent.

(2) Reduced form of Augmented matrix $[A \vdots B]_{12}=[AR \vdots C]$

(3) Existence of a solution

Theorem:

System AX=B has a solution iff A and [A$\vdots$B] have the same rank.

Rank(A)=rank([A$\vdots$B])

(4)Uniquence of a solution

Let A be n$\times$n. Then system AX=B have a unique solution iff

$\boxed{A_R=In \text{ or } rank(A)=n}$

$A_{n\times n}X=B \Rightarrow A_RX=C$

$\boxed{InX=C}$

4. Procedure for solving AX=B

Step 1. Find reduced matrix of [A$\vdots$B]$\rightarrow$[A$_R\vdots$C]

Step 2. Check if rank(A)=rank([A$\vdots$B]) for existence of solution.

Step3. Identify the dependent unknowns(作法同 system AX=0)

Step4. Find general solution by assigning independent unknowns any constant.

Ex. $$\begin{bmatrix} -1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=\begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Sol. $$[A\vdots B]_R=\begin{bmatrix} -1 & 1 & 3 & -2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$, reduced form [A$\vdots$B]

$$[A\vdots B]_R=\begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & 4 \end{bmatrix}=[A_R\vdots C]$$

rank(A)=rank(A$_R$)=2

rank([A$\vdots$B])=rank([A$_R\vdots$C])=2

$\therefore$ rank(A)=rank([A⋮B])

$\Rightarrow$ consistent system $\Rightarrow$ a solution exists.

From A_R, we have

X_1 and X_2= dependent unknowns

X_3 = independent unknowns.

Ex. $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Sol. $A_R X = C$

$\therefore X_1 = X_3 + 6$

$X_2 = -2X_3 + 4$

$$\Rightarrow X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_3 + 6 \\ -2X_3 + 4 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} X_3 + \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

Let $X_3 = \alpha$, general solution is

$$X = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

$= \quad H \quad + U_P$

H: general solution of AX=0

U_P: particular solution of AX=B

6-9 Matrix Inverses

Inverse of a square matrix. If AB=BA, then B is an inverse matrix of A.

(1)Definition of nonsingular matrix

A matrix that has an inverse is called nonsingular; otherwise called singular.

(2)Theorem of uniquence of an inverse

A nonsingular matrix has exactly one inverse.

[Proof] $\begin{cases} BA = I \\ CA = I \end{cases} \Rightarrow B=C$

B=BI=BAC=IC=C

(3) Theorem of existence of an inverse matrix $A n \times m$ is nonsingular iff $\boxed{A_R = I_n}$

2. Properties

Let A and B be nonsingular matrices

(1) $(AB)^{-1}=(BA)^{-1}$

(2) $(A^{-1})^{-1}=A$

(3) $(A^{t})^{-1}=(A^{-1})^{t}$

(4) $(In)^{-1}=In$

(5) AB and BA are singular if A and B are $n\times n$ matrix and either is singular.

3.Method for Finding Inverse

(1) Row operation method $[In \vdots A] \rightarrow [\Omega \vdots A_R]=[A^{-1} \vdots In]$

$\Omega A=A_R$, If $A_R=In$ $\therefore \Omega=A^{-1}$ $(\Omega A=In)$

Ex. $A=\begin{bmatrix}5 & -1\\ 6 & 8\end{bmatrix}$, $A^{-1}=?$

Sol. $[I_2 \vdots A]=\left[\begin{array}{cc|cc}1 & 0 & 5 & -1\\ 0 & 1 & 6 & 8\end{array}\right] \xrightarrow{row\_operation} \left[\begin{array}{cc|cc}\frac{8}{46} & \frac{1}{46} & 1 & 0\\ \frac{-6}{46} & \frac{5}{46} & 0 & 1\end{array}\right]=[\Omega \vdots A_R]=[A^{-1} \vdots I_2]$

$$\therefore A^{-1}=\Omega=\begin{bmatrix}\frac{8}{46} & \frac{1}{46}\\ \frac{-6}{46} & \frac{5}{46}\end{bmatrix}$$

(2) Determinants method (refer to section 7.7)

4. Use the Inverse to solve linear system

Let A be an $n\times n$ matrix. Then AX=B has a solution iff A is nonsingular.

$\because$ A is nonsingular $\therefore$ A has inverse A^{-1}

$\Rightarrow A^{-1}AX=A^{-1}B$ $\boxed{X=A^{-1}B}$

6.2 Elementary Row operations and Elementary Matrices AX=0

1. Elementary row operations

(1) type I : Interchange any two rows

(2) type II : multiply a row by a nonzero scalar

(3) type III : Add a scalar multiple of one row to another row

Ex: **AX=B**

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}=\begin{bmatrix}5\\ 6\end{bmatrix}$$

type I operation (row 1 $\longleftrightarrow$ row 2)

$$\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}=\begin{bmatrix}6\\ 5\end{bmatrix}$$

type II operation (-1/2 $\times$ row2)

$$\begin{bmatrix} 1 & 2 \\ -\frac{3}{2} & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

type III operation (-1/2 $\times$ row2 + row1)

$$\begin{bmatrix} -\frac{1}{2} & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

2. Elementary matrix

A matrix formed by performing an elementary row operation an identity matrix $\mathbf{I}_n$.

Ex. $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Type 1 row operation $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ elementary matrix

Type 2 row operation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ elementary matrix

(1) Theorem1

B=EA

Where **B**=matrix formed from A by elementary row operation

E=elementary matrix formed from I by the same row operation on A.

Ex.

$\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ type3 operation $-\frac{1}{2}\times$ row2$\rightarrow$ row1

Sol. $\mathbf{B}=\begin{bmatrix} -\frac{1}{2} & 0 \\ 3 & 4 \end{bmatrix}$

$\mathbf{I}_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \mathbf{E}=\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Sol. $\mathbf{EA}=\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} & 0 \\ 3 & 4 \end{bmatrix}=\mathbf{B}$

(2) Theorem2.

B=ΩA.

Where

B = matrix produced from **A** by any finite sequence of elementary row operations.

Ω = matrix of a product of elementary matrices formed form I by the same sequence of row operations on **A**.

Ex. $\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ $O_1=(\frac{1}{2}\times$ row2→ row1)on **A** to form $\mathbf{A}_1$.

$O_2=(6\times$ row1) on $\mathbf{A}_1$ to form $\mathbf{A}_2$.

$O_3=$(row1→ row2) on $\mathbf{A}_2$ to form **B**.

If **B=ΩA** then **B**=? **Ω**=?

Sol.

$$\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\xrightarrow{O_1}\mathbf{A}_1=\begin{bmatrix}-1/2 & 0\\ 3 & 4\end{bmatrix}\xrightarrow{O_2}\mathbf{A}_2=\begin{bmatrix}-3 & 0\\ 3 & 4\end{bmatrix}\xrightarrow{O_3}\mathbf{B}=\begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}$$

$$\mathbf{I}_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\xrightarrow{O_1}\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{bmatrix}\xrightarrow{O_2}\begin{bmatrix}6 & -3\\ 0 & 1\end{bmatrix}\xrightarrow{O_3}\begin{bmatrix}6 & -3\\ 6 & -2\end{bmatrix}=\mathbf{\Omega}$$

CHECK

$$\mathbf{\Omega A}=\begin{bmatrix}6 & -3\\ 6 & -2\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}=\mathbf{B}$$

$$\Omega=\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3$$

$$\begin{cases}\mathbf{I}\xrightarrow{O_1}\mathbf{E}_1\\ \mathbf{I}\xrightarrow{O_2}\mathbf{E}_2\\ \mathbf{I}\xrightarrow{O_3}\mathbf{E}_3\end{cases}$$

3. Row equivalence of matrices

Matrix A is row equivalent to matrix B if B can be obtained from A by a sequence of elementary row operations.

Theorems.

(1)Reflexive property $A\xrightarrow{\text{row equivalent}}A$

(2)Symmetric property $A\xleftrightarrow{\text{row equivalent}}B$

(3)Transitive property $A\xrightarrow{\text{r.e}}B\xrightarrow{\text{r.e}}C\Rightarrow A\xrightarrow{\text{r.e}}C$

6.3 The reduced Form of a Matrix

1. Zero row

All of its elements (entries) of a row of a matrix are Zero

2. Nonzero row

At least one element of a row is not Zero

3. Leading entry

The first nonzero element of a nonzero row (left$\rightarrow$right)

Ex: $\begin{bmatrix} 0 & 2 & 7 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

row	zero row	leading entry
1	X	2 (a_{12})
2	X	-2 (a_{22})
3	O	-
4	X	9 (a_{43})

4. Reduced Matrix

A matrix satisfies the following conditions

1. leading entry of nonzero row is 1
2. leading entry in column j , all other elements of column j are zero
3. if k a zero row , i a nonzero row , then i<k
4. if leading entry of row r_1 is in column c_1 , leading entry of row r_2 is in column c_2 , and $r_1<r_2$, then $c_1<c_2$

Ex $A = \begin{pmatrix} 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

row	zero row	Leading entry	Other elements of column in leading entry
1	X	1 (a_{11})	0
2	X	1 (a_{24})	0

$\rightarrow$A is reduced matrix

Ex $B = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

row	Zero row	Leading entry	Other entry of column of leading entry
1	X	1 (a_{12})	0
2	X	1 (a_{24})	0
3	X	1 (a_{35})	0
4	0	-	-
	(3)	(1) (4)	(2)

$\rightarrow$reduced matrix

Ex. C= $\begin{bmatrix} 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

row	Zero row	Leading entry	Other entry
1	x	1 (a_{12})	0
2	x	1 (a_{23})	5
3	x	1 (a_{34})	0
4	x	1 (a_{45})	0

Condition (2) not satisfied, NG. →not reduced matrix

$$C=\begin{bmatrix} 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow(-5\times row2+row1)\rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}=B$$

5.Theorems

(1)Equivalence to a Reduced Matrix

Every matrix **A** is row equivalent to a reduced matrix

(2)Reduced matrix A_R

A is a matrix, then there is exactly one reduced matrix A_R that is row equivalent to **A**. (Reduced matrix of A)

Ex. **A**= $\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ **A** is not a reduced matrix

$\mathbf{A}_R$=reduced form of **A**=?

Sol.

$$\mathbf{A}=\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \frac{1}{2}(\text{row } 1)\rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\to -2(\text{row }1)+(\text{ row }3)\to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\to(\text{row }2)+(\text{row }1)\text{，}-1(\text{row }2)+(\text{row }3)\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\to \frac{1}{3}(\text{row }3)\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to -1(\text{row }3)+(\text{row }2)\text{，}(\text{row }3)+(\text{row }1)\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3 = \mathbf{A}_R$$

(3) $\Omega\mathbf{A}=\mathbf{A}_\mathbf{R}$

Let A be an $n\times m$ matrix. Then there is an $n\times m$ matrix Ω such that $\Omega\mathbf{A}=\mathbf{A}_\mathbf{R}$.

How to find Ω?

Method

Augmented matrix $n\times(n+m)$ $[\mathbf{I}_n \vdots \mathbf{A}_{n\times m}]$

⇓

A sequence of row operations

⇓

Matrix $[\Omega \vdots \mathrm{A}_R]$

⇓

$\Omega\mathbf{A}=\mathbf{A}_\mathbf{R}$

Ex. 6.20

Given $A\begin{pmatrix} -3 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$

Find (1) $\mathbf{A}_\mathbf{R}$

(2) Ω such that $\Omega\mathbf{A}=\mathbf{A}_\mathbf{R}$

Sol.

$$[\mathbf{I}_2 \vdots \mathbf{A}_{2\times 4}] = \left\langle \begin{array}{cc|ccc} 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 4 & 2 & 1 \end{array} \right\rangle$$

$$\to -\frac{1}{3}(\text{row }1)\to \left\langle \begin{array}{cc|ccc} -1/3 & 0 & 1 & -1/3 & 0 \\ 0 & 1 & 4 & -2 & 1 \end{array} \right\rangle$$

$$\rightarrow \text{-4(row 1)+(row 2)} \rightarrow \left\langle \begin{array}{cc|ccc} -1/3 & 0 & 1 & -1/3 & 0 \\ 4/3 & 1 & 0 & -2/3 & 1 \end{array} \right\rangle$$

$$\rightarrow -\frac{2}{3}\text{(row 2)} \rightarrow \left\langle \begin{array}{cc|ccc} -1/3 & 0 & 1 & -1/3 & 0 \\ -2 & -3/2 & 0 & 1 & -3/2 \end{array} \right\rangle$$

$$\rightarrow \frac{1}{3}\text{(row 2)+(row 1)} \rightarrow \left\langle \begin{array}{cc|ccc} -1 & -1/2 & 1 & 0 & -1/2 \\ -2 & -3/2 & 0 & 1 & -3/2 \end{array} \right\rangle = [\Omega \vdots\ \mathbf{A_R}]$$

$$\Rightarrow \Omega = \begin{bmatrix} -1 & 1/2 \\ -2 & -3/2 \end{bmatrix} \text{ and } \mathbf{A_R} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \end{bmatrix}$$

check if $\Omega\mathbf{A} = \mathbf{A_R}$

6.4 Row and Column Spaces of a Matrix and Rank of a Matrix

1. Row space of a matrix

If A is any $n \times m$ matrix, the rows of A can be thought as vector in R^m. The set of all linear combinations of these row vectors forms the row space of A, and is a subspace of R^m.

EX:

$$\text{Matrix A} = \begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 15 & 8 \end{pmatrix}$$

sol: row space of A

α (1,-1,4,2)+ β (0,1,3,2)+ γ (3,-2,15,8)

Dimension of row space of A?

$\because$ row vectors of A are linearly dependent.

(3,-2,15,8)=3(1,-1,4,2)+ (0,1,3,2)

$\therefore$ row space of A is of the following form.

α (1,-1,4,2)+ β (0,1,3,2)

$\because$ vectors (1,-1,4,2) & (0,1,3,2) are linearly independent.

=> (1,-1,4,2) & (0,1,3,2) form the basis of row space of A.

=> Dimension=2.

2. column space of a matrix

If A is any $n \times m$ matrix, the column of A can be thought as vector in R^n. The set of all linear combinations of these column vectors forms the column space of A, and is a subspace of R^n.

$$A=\begin{pmatrix} a_{11} \cdots a_{1m} \\ \vdots \\ a_{n1} \quad a_{nm} \end{pmatrix}_{n\times m}$$

column vectors of A are

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} \ldots \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} \text{ in } R^n.$$

column space of A

$$C_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + C_2\begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + C_m\begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

EX：

$$A=\begin{pmatrix} -1 & 4 & 0 & 1 & 6 \\ -2 & 8 & 0 & 2 & 12 \end{pmatrix}$$

Column space of a

$$C_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 4 \\ 8 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ 0 \end{pmatrix} + C_4\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_5\begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$\because$ each column vector = $\alpha\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$\therefore$ column space of A $\qquad -C_1+4C_2+C_4+6C_5=0$

$\alpha\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ in R^2 $\qquad -2C_1+8C_2+2C_4+12C_5=0$

$\Rightarrow \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ form the basis $\Rightarrow$ Dimension=1

3. Theorems

(1) If A is a matrix，then the row and column spaces of A has the same dimension.

(2) If matrix B is formed from matrix A by an elementary row operation, then matrix A and B have the same row space.

4. Rank, rank(A) 秩

The rank of a matrix is the number of nonzero rows of the reduced from of a matrix

Rank(A) = rank(A_R)

= number of nonzero rows of A_R

EX：

$$A=\begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 15 & 8 \end{pmatrix} \qquad \text{rank(A)= ?}$$

sol： Find reduced from of A

$$A_R = \begin{pmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

∴ rank(A) = rank(A_R)=2

(1) Theorem

Let A be an n×n matrix. Then rank(A) = n iff A_R = I_n

(2) Lemma

If A is a reduced matrix, then rank of A equals the dimension of the row space of A.

rank(A) = rank(A_R)

= Dimension of row space of A

= Dimension of column space of A

6.5 solutions of homogeneous systems of linear equations

1. system of linear equations

Homogeneous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = 0 \\ \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = 0 \end{cases} \qquad \text{n equations in m unknowns}$$

In matrix from

AX= O

where

$$A = \lfloor a_{ij} \rfloor = \begin{bmatrix} a_{11} \cdots a_{1m} \\ a_{21} \quad \vdots \\ \vdots \quad \vdots \\ a_{n1} \quad a_{nm} \end{bmatrix} = \text{matrix off coefficients}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \text{matrix of unknowns。}$$

O = $n \times 1$ zero matrix

The reduced system of AX= 0 is

$A_R X = 0$

which has the same solution as AX= 0

1. Theorem

   Let A be n×m matrix。Then the liner homogeneous system AX= 0 has the same solution as the reduced system $A_R X= 0$。

2. Gauss – Jordan Reduction method for AX= 0

   (1) Find A_R

   (2) Determine the dependent and independent unknowns

   If column j of A_R contains the leading entry of some row，then x_j is independent unknowns 。

   (3) Solve for each dependent unknowns as a sum of constants times independent unknowns in each nonzero row of A_R.

   (4) Assign the independent unknowns any values and yield the general solution.

EX：

Given：linear system

$$\begin{cases} x_1 - 3x_2 + x_3 - 7x_4 + 4x_5 = 0 \\ x_1 + 2x_2 - 3x_3 = 0 \\ x_2 - 4x_3 + x_5 = 0 \end{cases}$$

Find ：general solution

sol :

In matrix from

$$\begin{bmatrix} 1 & -3 & 1 & -7 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$$\therefore\ A = \begin{bmatrix} 1 & -3 & 1 & -7 & 4 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 \end{bmatrix}$$

Step (1) Find A_R

$$\begin{pmatrix} 1 & -3 & 1 & -7 & 4 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 \end{pmatrix} \longrightarrow A_R = \begin{pmatrix} 1 & 0 & 0 & -\frac{35}{16} & \frac{13}{16} \\ 0 & 1 & 0 & \frac{28}{16} & -\frac{20}{16} \\ 0 & 0 & 1 & \frac{7}{16} & -\frac{9}{16} \end{pmatrix}$$

(2) Determine dependent and independent unknowns columns 1,2,3,of A_R contain the leading entries

Result to dependent unknowns : X_1, X_2, X_3

Independent unknowns: X_4, X_5

(3)Solve reduced system $A_R X=0$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{35}{16} & \frac{13}{16} \\ 0 & 1 & 0 & \frac{28}{16} & -\frac{20}{16} \\ 0 & 0 & 1 & \frac{7}{16} & -\frac{9}{16} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} = 0$$

$X_1 =$ 35/16x4 – 13/16x5

$X_2 =$ -28/16x4 + 20/16x5

$X_3 =$ -7/16x4 + 9/16x5

(4) Assign any value to independent unknowns:

$X_4 = 16\alpha$, $X_5 = 16\beta$, α, β : arbitrary constants

General solution

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} = \alpha \begin{pmatrix} 35 \\ -28 \\ -7 \\ 16 \\ 0 \end{pmatrix} \beta \begin{pmatrix} -13 \\ 20 \\ 9 \\ 0 \\ 16 \end{pmatrix}$$

Rank(A) = rank(RA)

=3

=number of dependent unknowns

Back to column space

$$A=\begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 15 & 8 \end{pmatrix}$$

Column space of A

$$C_1\begin{pmatrix}1\\0\\3\end{pmatrix}+C_2\begin{pmatrix}-1\\1\\-2\end{pmatrix}+C_3\begin{pmatrix}4\\3\\15\end{pmatrix}+C_4\begin{pmatrix}2\\2\\8\end{pmatrix}$$

Let

$$C_1\begin{pmatrix}1\\0\\3\end{pmatrix}+C_2\begin{pmatrix}-1\\1\\-2\end{pmatrix}+C_3\begin{pmatrix}4\\3\\15\end{pmatrix}+C_4\begin{pmatrix}2\\2\\8\end{pmatrix}=0$$

$$\begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 15 & 8 \end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix}=0$$

Homogeneous linear equation Reduced system is

$$\begin{pmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix}=0$$

Result to $C_1 = -7C_3 - 4C_4$

$C_2 = -3C_3 - 2C_4$

$$C_1\vec{F}_1 + C_2\vec{F}_2 = (-7C_3 - 4C_4)\vec{F}_1 + (-3C_3 - 2C_4)\vec{F}_2$$
$$= -C_3(7\vec{F}_1 + 3\vec{F}_2) - C_4(4\vec{F}_1 + 2\vec{F}_2) = -C_3\vec{F}_3 - C_4\vec{F}_4$$

Result to $\vec{F}_1, \vec{F}_2, \vec{F}_3, \vec{F}_4$ linearly dependent

$$\vec{F}_3 = 7\vec{F}_1 + 3\vec{F}_2$$
$$\vec{F}_4 = 4\vec{F}_1 + 2\vec{F}_2$$

Column space of A is

$$\alpha\begin{pmatrix}1\\0\\3\end{pmatrix}+\beta\begin{pmatrix}-1\\1\\-2\end{pmatrix}$$

$\begin{pmatrix}1\\0\\3\end{pmatrix},\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ form the basis of column space of, Dimension of column space of A=2

6.6 The solution space of AX=0

1. solution space

A is a n×m matrix and AX=0

Any solution (mx1 column matrix) of AX=0can be thought of as a vector in RM.

Then solutions of AX=0 are a subspace of RM and called the solution space of system AX=0

pf:

The set of solution of AX=0

$AX_1=0$ $AX_2=0$

$A(X_1+X_2)=AX_1+AX_2=0$

A sum of solutions is a solution

$A(\alpha X_1)= \alpha (AX_2)=0$

Scalar multiplication of a solution is a solution.

2.Dimension of solution space

Let A be an n×m matrix of real number.

Then the solution space of AX=0 has dimension m=rank(A)

pf:

general solution of AX=0=linear combination of linearly independent vectors..

No. of linearly independent vectors

= No. of arbitrary constants

= m-rank(A)

No. of linearly independent vectors

= a basis of solution space

= dimension of solution space

Ex. System AX=0 $\begin{bmatrix} -1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \\ -3 & 1 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$=0, the solution space=?

Sol. $A_R=\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{9}{8} \\ 0 & 1 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 1 & \frac{-1}{4} \end{bmatrix}$

$\because m=5$ rank(A)=rank(A_R)=4

$\therefore$ Dimension of solution space of AX=0

m-rank(A)=5-4=1

General solution is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}=\begin{bmatrix} \frac{-9}{8} \\ \frac{5}{8} \\ \frac{9}{8} \\ \frac{-1}{4} \\ 1 \end{bmatrix}$$ One arbitrary constant.

3. Trivial solution

The solution $X_1= X_2=\ldots\ldots =X_n=0$ is the trivial solution of a homogeneous system AX=0.

4. Existence of nontrivial solution

(1)Theorem

Let A be $n\times m$ matrix. The system AX=0 has a nontrivial solution iff m>rank(A).

$\because$ Dimension of solution space of AX=0=m-rank(A)>0

Then system AX=0 has a nonzero solution!

(2)Corollary

A. AX=0 always has a nontrivial solution if the number of unknowns exceed the number of equations.

A. $A_{n\times m}X_{m\times 1}=0$, n is the number of equations; m is the number of unknowns.

$\because n<m$ and $\text{rank}(A)\le n$. $\therefore m-\text{rank}(A)\ge m-n>0$.

B. A is an n×m matrix of real number. Then the system AX=0 has only the trivial solution iff $\boxed{A_R=I_n}$

$\because A_R=I_n$

$\therefore$ rank(A)=rank(A_R)=n, and m=n.

$\therefore$ m-rank(A)n-n=0

$\Rightarrow$ System AX=0 has only zero solution.

6-7 Nonhomogeneous Systems of Linear Equations

1. Nonhomogeneous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...+ a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...+ a_{2m}x_m = b_2 \\ .... \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ...+ a_{nm}x_m = b_n \end{cases}$$

I_n matrix from AX=B

When A=[a_{ij}]=n×m coefficient matrix

X=[X_j]=m×1 unknowns

B=[b_i]=n×1 matrix (at least one of which is nonzero).

Augmented matrix of system AX=B

$$[A\vdots B]=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1m} & b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nm} & b_n \end{bmatrix}$$

2.Structure of solution of AX=B

(1)Theorem

Let Up be a solution of AX=B, then every solution of AX=Bi the form of U_p+H. in which H is a solution of AX=0.

[Proof]

Assume $\boxed{AW=B}$ $\because$ $\boxed{AU_p=B}$

$\therefore$ AW=AU_p=A(W-U_P)=B-B=0

$\therefore$ W-U_P=H is the solution of AX=0

$\therefore$ W=U_P+H

(2) Remark on U_P and H

A. U_P=particular solution of AX=0

B. H=general solution of AX=0

C. U_P+H=general solution of AX=0

3. Existence and Uniquences of solutions

(1) Consistent system of equation

Definition:

AX=B is said to be consistent if there exists a solution, otherwise the system is inconsistent.

(2) Reduced form of Augmented matrix $[A\vdots B]_{12}=[AR\vdots C]$

(3) Existence of a solution

Theorem:

System AX=B has a solution iff A and [A⋮B] have the same rank.

Rank(A)=rank([A⋮B])

(4)Uniquence of a solution

Let A be n×n. Then system AX=B have a unique solution iff

A_R=In or rank(A)=n

$A_{n\times n}X=B \Rightarrow A_RX=C$

InX=C

4. Procedure for solving AX=B

Step 1. Find reduced matrix of [A⋮B]→[A_R⋮C]

Step 2. Check if rank(A)=rank([A⋮B]) for existence of solution.

Step3. Identify the dependent unknowns(作法同 system AX=0)

Step4. Find general solution by assigning independent unknowns any constant.

Ex. $\begin{bmatrix} -1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=\begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Sol. $[A\vdots B]_R=\begin{bmatrix} -1 & 1 & 3 & -2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$, reduced form [A⋮B]

$[A\vdots B]_R=\begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & 4 \end{bmatrix}=[A_R\vdots C]$

rank(A)=rank(A_R)=2

rank([A⋮B])=rank([A_R⋮C])=2

∴ rank(A)=rank([A⋮B])

⇒consistent system ⇒a solution exists.

From A_R, we have

X_1 and X_2= dependent unknowns

X_3 = independent unknowns.

Ex. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Sol. $A_R X = C$

$\therefore X_1 = X_3 + 6$

$X_2 = -2X_3 + 4$

$$\Rightarrow X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_3 + 6 \\ -2X_3 + 4 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} X_3 + \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

Let $X_3 = \alpha$, general solution is

$$X = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

$= \quad H \quad + U_P$

H: general solution of AX=0

U_P: particular solution of AX=B

6-9 Matrix Inverses

Inverse of a square matrix. If AB=BA, then B is an inverse matrix of A.

(1)Definition of nonsingular matrix

A matrix that has an inverse is called nonsingular; otherwise called singular.

(2)Theorem of uniquence of an inverse

A nonsingular matrix has exactly one inverse.

[Proof] $\begin{cases} BA = I \\ CA = I \end{cases} \Rightarrow B = C$

B=BI=BAC=IC=C

(3) Theorem of existence of an inverse matrix $A n \times m$ is nonsingular iff $\boxed{A_R = I_n}$

2. Properties

Let A and B be nonsingular matrices

(1) $(AB)^{-1} = (BA)^{-1}$

(2) $(A^{-1})^{-1} = A$

(3) $(A^t)^{-1} = (A^{-1})^t$

(4) $(I_n)^{-1} = I_n$

(5) AB and BA are singular if A and B are $n \times n$ matrix and either is singular.

3.Method for Finding Inverse

(2) Row operation method $[I_n \vdots A] \rightarrow [\Omega \vdots A_R] = [A^{-1} \vdots I_n]$

$\Omega A = A_R$, If $A_R = I_n$ $\therefore \Omega = A^{-1}$ $(\Omega A = I_n)$

Ex. $A = \begin{bmatrix} 5 & -1 \\ 6 & 8 \end{bmatrix}$, $A^{-1} = ?$

Sol. $[I_2 \vdots A] = \left[\begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 1 & 6 & 8 \end{array}\right] \xrightarrow{row\_operation} \left[\begin{array}{cc|cc} \frac{8}{46} & \frac{1}{46} & 1 & 0 \\ \frac{-6}{46} & \frac{5}{46} & 0 & 1 \end{array}\right] = [\Omega \vdots A_R] = [A^{-1} \vdots I_2]$

$$\therefore A^{-1} = \Omega = \begin{bmatrix} \frac{8}{46} & \frac{1}{46} \\ \frac{-6}{46} & \frac{5}{46} \end{bmatrix}$$

(2) Determinants method (refer to section 7.7)

4. Use the Inverse to solve linear system

Let A be an $n \times n$ matrix. Then AX=B has a solution iff A is nonsingular.

$\because$ A is nonsingular $\therefore$ A has inverse A^{-1}

$\Rightarrow A^{-1}AX = A^{-1}B$ $\boxed{X = A^{-1}B}$