CHAPTER 5

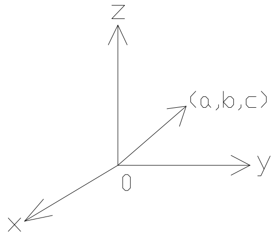
Vectors and Vector Space

5.1 Algebra and Geometry of Vectors

1. Vector

   An ordered triple (a, b, c), where a, b, c are real numbers.

   Symbol: $a, B, \vec{a}, \vec{b}$



   A magnitude and a direction.

2. Norm of a vector (a, b, c)

   Norm $= \|(a,b,c)\|$

   $= \sqrt{a^2 + b^2 + c^2}$

   = magnitude

3. Scalar multiplication

   Product of a scalar and vector

   $\vec{F} = (a,b,c)$, scalar α

   $\alpha\vec{F} = \alpha(a,b,c)$

   $= (\alpha a, \alpha b, \alpha c)$

4. Sum of vectors (Addition)

   $\vec{F} = (a_1, b_1, c_1)$

   $\vec{G} = (a_2, b_2, c_2)$

   $\vec{F} + \vec{G} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

5. Algebra of Vectors

   (1) $\vec{F} + \vec{G} = \vec{G} + \vec{F}$ (Commutative law)

   $\vec{F} + \vec{G} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

   $= (a_2 + a_1, b_2 + b_1, c_2 + c_1)$

   $= (a_2, b_2, c_2) + (a_1, b_1, c_1)$

   $= \vec{G} + \vec{F}$

   (2) $(\vec{F} + \vec{G}) + \vec{H} = \vec{F} + (\vec{G} + \vec{H})$ (Associative law)

   (3) $\vec{F} + (0,0,0) = \vec{F}$

   $(0,0,0) = \vec{0}$ (Additive identity)

   (4) $\alpha(\vec{F} + \vec{G}) = \alpha\vec{F} + \alpha\vec{G}$

   (5) $\alpha\beta\vec{F} = \alpha(\beta\vec{F})$

   (6) $(\alpha + \beta)\vec{F} = a\vec{F} + \beta\vec{F}$

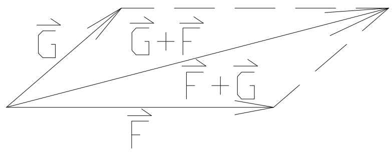
6. Norm of scalar multiplication

(1) $\|\alpha\vec{F}\| = |\alpha|\|\vec{F}\|$

(2) $\|\vec{F}\| = 0$ Iff $\vec{F} = 0$

$$\|\vec{F}\| = \sqrt{a^2 + b^2 + c^2} = 0$$
$$\Rightarrow a = b = c = 0$$
$$\Rightarrow \vec{F} = (0,0,0) = \vec{0}$$

7. Parallelogram Law for Vector Addition

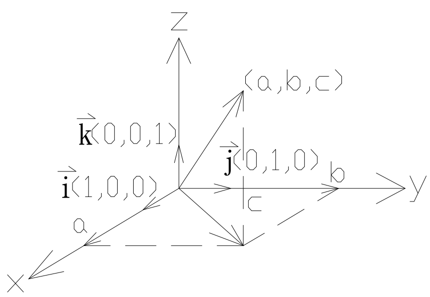


$\vec{F} + \vec{G} = \vec{G} + \vec{F}$

8. Standard representation of Vector or Cartesian Vector form

$\vec{F} = a\vec{i} + b\vec{j} + c\vec{k}$

where $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$ are unit vectors, which represent the directions of x, y, z axes, respectively.



5.2 Dot product

1. Dot product (Scalar product, inner product)

The dot product of vectors $\vec{F}$ and $\vec{G}$ is the scalar.

$\vec{F} \bullet \vec{G} = a_1a_2 + b_1b_2 + c_1c_2$

2. Properties of dot product

(1) $\vec{F} \bullet \vec{G} = \vec{G} \bullet \vec{F}$ (Commutative law)

$$\vec{F} \bullet \vec{G} = a_1a_2 + b_1b_2 + c_1c_2$$
$$= a_1a_2 + b_1b_2 + c_1c_2$$
$$= \vec{G} \bullet \vec{F}$$

(2) $\left(\vec{F} + \vec{H}\right) \bullet \vec{G} = \left(\vec{F} \bullet \vec{G}\right) + \left(\vec{H} \bullet \vec{G}\right)$ (Distributive law)

(3) $\alpha\left(\vec{F} \bullet \vec{G}\right) = \left(\alpha\vec{F} \bullet \vec{G}\right) = \vec{F} \bullet \left(\alpha\vec{G}\right)$ α ：scalar

(4) $\vec{F} \bullet \vec{F} = \|\vec{F}\|^2$

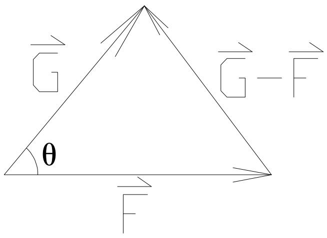
$$\vec{F} \bullet \vec{F} = a_1^{\ 2} + b_1^{\ 2} + c_1^{\ 2}$$

$$= \left(\sqrt{a_1^{\ 2} + b_1^{\ 2} + c_1^{\ 2}}\right)^2$$

$$= \|\vec{F}\|^2$$

(5) $\vec{F} \bullet \vec{F} = 0$ iff $\vec{F} = \vec{0}$

3. Geometrical interpretation of dot product



$\vec{F} = (a_1, b_1, c_1)$

$\vec{G} = (a_2, b_2, c_2)$

(1) Determine the angle between vectors (Law of cosine)

$$\|\vec{G}\|^2 + \|\vec{F}\|^2 - 2\|\vec{G}\|\|\vec{F}\|\cos\theta = \|\vec{G} - \vec{F}\|^2$$

$$\|\vec{G} - \vec{F}\|^2 = (\vec{G} - \vec{F}) \bullet (\vec{G} - \vec{F}) = \|\vec{G}\|^2 - 2\vec{F}\vec{G} + \|\vec{F}\|^2$$

$$\therefore -2\|\vec{G}\|\|\vec{F}\|\cos\theta = -2\vec{F} \bullet \vec{G}$$

$$\Rightarrow \cos\theta = \frac{\vec{F} \bullet \vec{G}}{\|\vec{G}\|\|\vec{F}\|}$$

$\therefore \theta$ = angle between the two vectors

$$= \cos^{-1}\left(\frac{\vec{F} \bullet \vec{G}}{\|\vec{G}\|\|\vec{F}\|}\right)$$

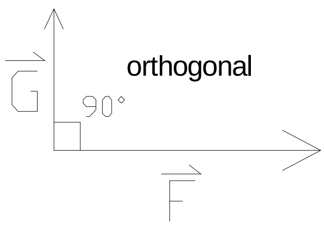
(2) Orthogonal vectors

A. Two nonzero vectors $\vec{F}$ and $\vec{G}$ are orthogonal iff $\vec{F} \bullet \vec{G} = 0$

$$\vec{F} \bullet \vec{G} = \|\vec{F}\|\|\vec{G}\|\cos\theta = 0$$

$$\because \|\vec{F}\| \neq 0, \|\vec{G}\| \neq 0$$

$$\therefore \cos\theta = 0, \quad \theta = 90°$$



B. $\vec{0} \bullet \vec{F} = 0$ for every $\vec{F}$

$\Rightarrow$ zero vector $\vec{0}$ orthogonal to every vector.

Ex. Two lines

$L_1 : (2-4t, 6+t, 3t)$

$L_2 : (-2+p, 7+2p, 3-4p)$

$L_1 \perp L_2$?

Sol: point of intersection of L_1 & L_2

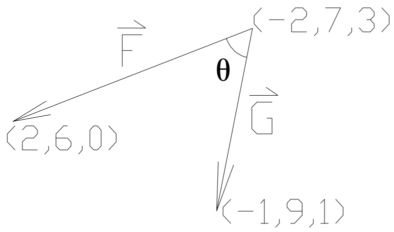
$t = 0 \rightarrow (2,6,0)$

$\begin{cases} t = 1 \rightarrow (-2,7,3) \\ p = 0 \rightarrow (-2,7,3) \end{cases}$

$p = 1 \rightarrow (-1,9,-1)$

$\vec{F} = (4,-1,-3)$

$\vec{G} = (1,2,-4)$



$\therefore \vec{F} \bullet \vec{G} = 4 - 2 + 12 = 14 \neq 0$

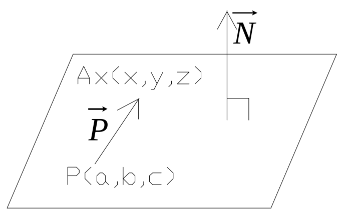
$\therefore L_1$ is not $\perp L_2$

(3) Equation of a plane

$\vec{P} = (x, y, z) - (a, b, c) = (x-a)\vec{i} + (y-b)\vec{j} + (z-c)\vec{k}$

$\vec{N} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$

$\because \vec{N} \perp Plane \quad \pi$



$\therefore$ any vector on plane π is orthogonal to its normal vector $\vec{N}$

$\therefore \vec{P} \cdot \vec{N} = 0$

$\therefore n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$

$\Rightarrow n_1x + n_2y + n_3z = n_1a + n_2b + n_3c$

4. Cauchy-Schwarz inequality

$\left|\vec{F} \cdot \vec{G}\right| \leq \left\|\vec{F}\right\| \left\|\vec{G}\right\|$

$$\because \cos\theta = \frac{\overrightarrow{F}\cdot\overrightarrow{G}}{\|\overrightarrow{F}\|\|\overrightarrow{G}\|}$$

$$-1 \le \cos\theta \le 1$$

$$\therefore -\|\overrightarrow{F}\|\|\overrightarrow{G}\| \le \overrightarrow{F}\cdot\overrightarrow{G} \le \|\overrightarrow{F}\|\|\overrightarrow{G}\|$$

5.3 The Cross Product

1. Cross product

The cross product of vectors $\overrightarrow{F}$ and $\overrightarrow{G}$ is the vector

$$\overrightarrow{F}\times\overrightarrow{G} = (b_1c_2 - b_2c_1)\vec{i} + (a_2c_1 - a_1c_2)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

(i) $$\overrightarrow{F}\times\overrightarrow{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\vec{k}$$

(ii) $$\overrightarrow{F}\times\overrightarrow{G} = (a_1\vec{i} + b_1\vec{j} + c_1\vec{k})\times(a_2\vec{i} + b_2\vec{j} + c_2\vec{k})$$

$$= a_1a_2\vec{i}\times\vec{i} + a_1b_2\vec{i}\times\vec{j} + a_1c_2\vec{i}\times\vec{k}$$

$$+ b_1a_2\vec{j}\times\vec{i} + b_1b_2\vec{j}\times\vec{j} + b_1c_2\vec{j}\times\vec{k}$$

$$+ c_1a_2\vec{k}\times\vec{i} + c_1b_2\vec{k}\times\vec{j} + c_1c_2\vec{k}\times\vec{k}$$

2. Properties of cross product

(1) $\overrightarrow{F}\times\overrightarrow{G} = -\overrightarrow{G}\times\overrightarrow{F}$ (Anti-commutativity)

(2) $\overrightarrow{F}\times\overrightarrow{G}$ orthogonal to both $\overrightarrow{F}$ and $\overrightarrow{G}$

pf. $$\overrightarrow{F}\cdot(\overrightarrow{F}\times\overrightarrow{G}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\overrightarrow{G}\cdot(\overrightarrow{F}\times\overrightarrow{G}) = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

(3) $$\|\overrightarrow{F}\times\overrightarrow{G}\| = \|\overrightarrow{F}\|\|\overrightarrow{G}\|\sin\theta$$

$$\|\overrightarrow{F}\times\overrightarrow{G}\|^2 = \|\overrightarrow{F}\|^2\|\overrightarrow{G}\|^2 - (\overrightarrow{F}\cdot\overrightarrow{G})^2$$

$$= \|\vec{F}\|^2\|\vec{G}\|^2 - \|\vec{F}\|^2\|\vec{G}\|^2\cos^2\theta = \|\vec{F}\|^2\|\vec{G}\|^2\sin^2\theta$$

$\because 0 \le \theta \le \pi$

(4) $\vec{F}\times\vec{G}=0$ iff $\vec{F}\,//\,\vec{G}$

$\|\vec{F}\times\vec{G}\| = \|\vec{F}\|\|\vec{G}\|\sin\theta \qquad \|\vec{F}\|\neq 0, \|\vec{G}\|\neq 0$

$\therefore \sin\theta = 0, \theta = 0^\circ \; or \; \pi$

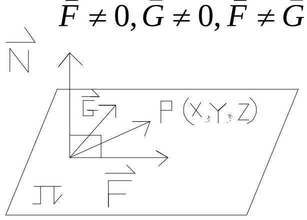
$\therefore \vec{F}\,//\,\vec{G}$

(5) $\vec{F}\times(\vec{G}+\vec{H}) = \vec{F}\times\vec{G}+\vec{F}\times\vec{H}$ (Associativity)

(6) $\alpha(\vec{F}\times\vec{G}) = (\alpha\vec{F})\times\vec{G} = \vec{F}\times(\alpha\vec{G})$

3. Plane determined by $\vec{F}$ and $\vec{G}$

$\vec{F}\neq 0, \vec{G}\neq 0, \vec{F}\neq\vec{G}$



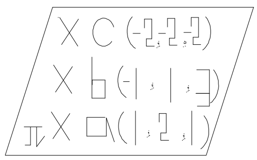
Normal vector $\vec{N} = \vec{F}\times\vec{G}$

Equation of plane

$\vec{N}\cdot\vec{P} = 0 = (\vec{F}\times\vec{G})\cdot\vec{P}$

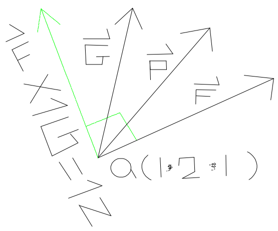
Ex.5.11

Given:



Find: equation of plane π

Sol:



$\vec{F} = (-2,-1,2)$

$\vec{G} = (-3,-4,-3)$

$\vec{N} = \vec{F}\times\vec{G}$

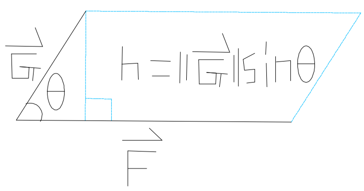
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & 2 \\ -3 & -4 & -3 \end{vmatrix} = 11\vec{i} - 12\vec{j} + 5\vec{k}$$

$\therefore$ Equation of plane π

$\vec{N}\cdot\vec{P} = (11,-12,5)\cdot(x-1, y-2, z-1) = 0$

$\Rightarrow 11x - 12y + 5z = -8$

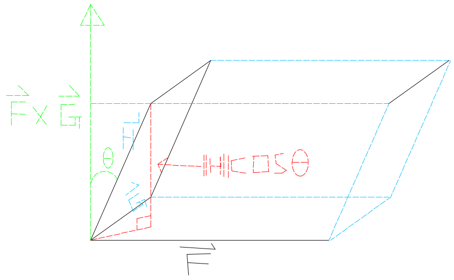
4. Area of parallelogram



Area Formula of geometry

Area $= \|\vec{F}\|h = \|\vec{F}\|\|\vec{G}\|\sin\theta = \|\vec{F}\times\vec{G}\|$

5. Volume of parallelepiped



$$V = \text{area of base} \times \text{altitude}$$

$$= \left\|\vec{F} \times \vec{G}\right\| \left\|\vec{H}\right\| \cos\theta$$

$$= \left|\vec{F} \cdot (\vec{F} \times \vec{G})\right|$$

6. Scalar triple product

$$[\vec{H}, \vec{F}, \vec{G}] = \vec{H} \cdot (\vec{F} \times \vec{G}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Properties:

(1) $[\vec{H}, \vec{F}, \vec{G}] = [\vec{G}, \vec{H}, \vec{F}] = [\vec{F}, \vec{G}, \vec{H}]$

(2) $[\vec{H}, \vec{F}, \vec{G}] = -[\vec{H}, \vec{G}, \vec{F}]$

5.4 Vector space R^n

1. n-vector

An n-vector is an ordered n-tuple $(x_1, x_2, x_3 \cdots\cdots x_n)$

Where x_j is a real number $(n \geq 2)$

The set of all n-vectors is denoted R^n

R^2: set of order pair (x, y)

R^3: set of all 3-vectors

R^4: set of all 4-vector (u, v, w, s)

2. Algebra of R^n (vector operation)

(1) Addition of n-vectors

$$(x_1, x_2, x_3 \cdots\cdots x_n) + (y_1, y_2, y_3 \cdots\cdots y_n) = (x_1 + y_1, x_2 + y_2 \cdots\cdots x_n + y_n)$$

(2) Scalar multiplication

$$\alpha (x_1, x_2, x_3 \cdots\cdots x_n) = (\alpha x_1, \alpha x_2, \alpha x_3 \cdots\cdots \alpha x_n)$$

3. Properties of n-vectors

(1) $\vec{F} + \vec{G} = \vec{G} + \vec{F}$

(2) $\vec{F} + (\vec{G} + \vec{H}) = (\vec{F} + \vec{G}) + \vec{H}$

(3) $\vec{F} + \vec{0} = \vec{F}$

(4) $(\alpha + \beta)\vec{F} = \alpha\vec{F} + \beta\vec{F}$

(5) $(\alpha\beta)\vec{F} = \alpha(\beta\vec{F})$

(6) $\alpha(\vec{F}+\vec{G})=\alpha\vec{F}+\alpha\vec{G}$

(7) $\alpha\vec{0}=\vec{0}$

Because of these properties of addition of n-vectors and multiplication by scalar, R^n is called <u>vector space</u> or <u>real vector space</u>.

4. Norm of n-vector

$$\|(x_1, x_2 \cdots\cdots x_n)\| = \sqrt{x_1{}^2 + x_2{}^2 + \cdots\cdots x_n{}^2} = \sqrt{\sum_{j=1}^{n} x_j{}^2}$$

5. Dot product of n-vectors

$$(x_1, x_2, x_3 \cdots\cdots x_n) \cdot (y_1, y_2, y_3 \cdots\cdots y_n) = x_1 y_1 + x_2 y_2 + \cdots\cdots x_n y_n$$

A. properties of dot product of n-vectors

(1) $\vec{F}\cdot\vec{G}=\vec{G}\cdot\vec{F}$

(2) $(\vec{F}+\vec{H})\cdot\vec{G}=\vec{F}\cdot\vec{G}+\vec{H}\cdot\vec{G}$

(3) $\alpha(\vec{F}\cdot\vec{G})=(\alpha\vec{F})\cdot\vec{G}=\vec{F}\cdot(\alpha\vec{G})$

(4) $\vec{F}\cdot\vec{F}=\|\vec{F}\|^2$

(5) $\vec{F}\cdot\vec{F}=0$ iff $\vec{F}=0$

Notes:

(1) No parallegram law in R^n

(2) No higher dimensional analogue of the cross product (n>3)

(3) No general version of the law of cosine in R^n

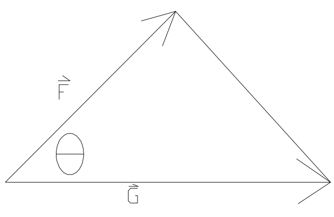
B. Cauchy-Schwarz inequality

$|\vec{F}\cdot\vec{G}| \le \|\vec{F}\|\|\vec{G}\|$ (Purely computational proof) Refer to P.224

C. Angle θ between n-vectors $\vec{F}$ and $\vec{G}$

$\vec{F}\ne 0, \vec{G}\ne 0$

$$\cos\theta = \frac{\vec{F}\cdot\vec{G}}{\|\vec{F}\|\|\vec{G}\|} \qquad 0\le\theta\le\pi$$



$\vec{F}$ orthogonal to $\vec{G}$ iff $\vec{F}\cdot\vec{G}=0$

6. Standard representation of vectors in R^n

$\vec{F}=(x_1, x_2, \cdots, x_n)$

$$= x_1\vec{e_1} + x_2\vec{e_2} + \cdots + x_n\vec{e_n} = \sum_{j=1}^{n} x_j\vec{e_j}$$

where $\vec{e_1}=(1,0,0,\cdots,0)$, $\vec{e_2}=(0,1,0,\cdots,0)$, ...

And $\overrightarrow{e_n} = (0,0,0,\cdots,1)$ are mutually orthogonal unit vectors which define the n directions of $\vec{F}$

$$\overrightarrow{e_i} \cdot \overrightarrow{e_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

7. Subspace of n-space

A set s of vectors in R^n is called a subspace of R^n

If (1) $\vec{0} \in S$

(2) $(\vec{F} + \vec{G}) \in S$ when $\vec{F} \,\&\, \vec{G} \in S$

(3) $\alpha\vec{F} \in S$ α : real number

Ex. Given: S = $\{\vec{F} \mid \vec{F} \in R^n, \|\vec{F}\| = 1\}$

Find: Is S a subspace of R^n

$$\text{Sol: } \begin{cases} (1)\ \|0\| = 0 \neq 1 & \vec{0} \in S \\ (2)\ \left\|\vec{F_1} + \vec{F_2}\right\| \neq 1 & \vec{F_1} + \vec{F_2} \notin S \\ (3)\ \left\|\alpha\vec{F}\right\| = |\alpha|\left\|\vec{F}\right\| = |\alpha| \neq 1 & \alpha\vec{F} \in S \end{cases}$$

$\therefore$ S is not a subspace of R^n

Ex. Given: k = $\{\vec{F} \mid \vec{F} \in R^4, \vec{F} = \alpha(\text{-}1,4,2,0)\}$

Find: Is k a subspace of R^4 ?

Sol: (1) $\vec{0} = 0 \cdot (\text{-}1,4,2,0) = 0$ $\vec{0} \in k$

(2) $\overrightarrow{F_1} + \overrightarrow{F_2} = \alpha_1(\text{-}1,4,2,0) + \alpha_2(\text{-}1,4,2,0)$

$= (\alpha_1 + \alpha_2)(\text{-}1,4,2,0) = \alpha(\text{-}1,4,2,0)$

$\overrightarrow{F_1} + \overrightarrow{F_2} \in k$

(3) $\beta\overrightarrow{F_1} = \beta\alpha_1(\text{-}1,4,2,0) = \alpha(\text{-}1,4,2,0)$

$\beta\overrightarrow{F_1} \in k$

$\therefore$ k is a subspace of R^4

5.5 Linear Independence and Dimension in R^n

1. Linear combination of n-vectors

$c_1\overrightarrow{F_1}+c_2\overrightarrow{F_2}+...+c_k\overrightarrow{F_k}$

Here $\begin{cases}\overrightarrow{F_k}\in R^n\\ k=1,2,...\\ C_j=scalar\end{cases}$

EX：

$\overrightarrow{F_1}=(-2,4,1,0)\quad \overrightarrow{F_2}=(1,1,-1,7)\quad \overrightarrow{F_3}=(8,0,0,0)\qquad \in R^4$

Linear combination?

$c_1(-2,4,1,0)+c_2(1,1,-1,7)+c_3(8,0,0,0)$

2. Linear Dependence of n-vectors

$\overrightarrow{F_1},\overrightarrow{F_2},...\overrightarrow{F_k}$ are linear dependence

if there are real number C_k not all zero such that

$c_1\overrightarrow{F_1}+c_2\overrightarrow{F_2}+...+c_k\overrightarrow{F_k}=0$

EX：

$\overrightarrow{F_1}=(-1,2,0,0,0)\quad \overrightarrow{F_2}=(3,-6,0,0,0)\qquad \in R^5$

linearly dependent？

Sol： Let $c_1\overrightarrow{F_1}+c_2\overrightarrow{F_2}+...+c_k\overrightarrow{F_k}=0$

$\begin{cases}-c_1+3c_3=0\\ 2c_1-6c_2=0\end{cases}\quad c_1=3c_2$

Take $c_1=3$ $c_2=1$ not all zeros

$\therefore \overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are linearly dependence

3. Linear Independence of n-vectors

$\overrightarrow{F_1},\overrightarrow{F_2},...\overrightarrow{F_k}$ are linear independence iff $c_1\overrightarrow{F_1}+c_2\overrightarrow{F_2}+...+c_k\overrightarrow{F_k}=0$

can hold only if all $C_k=0$

EX：

$$\begin{cases}\overrightarrow{F_1} = (1,0,0) \\ \overrightarrow{F_2} = (0,1,0)\end{cases} \in R^3$$

Sol：Let $c_1\overrightarrow{F_1} + c_2\overrightarrow{F_2} = 0 \Rightarrow C_1 = C_2 = 0 \Rightarrow \overrightarrow{F_1} and \overrightarrow{F_2}$ linearly independent

4. Condition for linearly independent

$$(1)\begin{cases}F_{11}C_1 + F_{21}C_2 + \cdots + F_{k1}C_k = 0 \\ F_{21}C_1 + F_{22}C_2 + \cdots + F_{k2}C_k = 0 \\ \vdots \\ F_{1n}C_1 + F_{2n}C_2 + \cdots + F_{kn}C_k = 0\end{cases}$$

Solve above system of equation to obtain $C_1, C_2 \cdots C_k$

If $C_1 = C_2 = \cdots = C_k = 0$ then are $\vec{F}_1, \vec{F}_2 \cdots \vec{F}_k$ linearly independent, otherwise linearly dependent.

(2) $\vec{F}_1, \vec{F}_2 \cdots \vec{F}_k$ vectors in R^n are linearly dependent if

I . No $\vec{F}_j$ is zero vector, and

II. If the j^{th} component of one vector is its first nonzero component, then all other vectors have component j equal to zero.

Ex :

$$\begin{cases}\vec{F}_1 = (0,4,0,0,2) \\ \vec{F}_2 = (0,0,6,0,-5) \\ \vec{F}_3 = (0,0,0,-4,12)\end{cases}$$

Sol :

$$C_1\vec{F}_1 + C_2\vec{F}_2 + C_3\vec{F}_3 = 0$$

$$\begin{cases}4C_1 + 0C_2 + C_3 = 0 \\ 0C_1 + 6C_2 + 0C_3 = 0 \Rightarrow C_1 = C_2 = C_3 = 0 \\ 0C_1 + 0C_2 - 4C_3 = 0 \Rightarrow linearly \cdot independent\end{cases}$$

(3)Let $\vec{F}_1, \vec{F}_2, \cdots \vec{F}_k$ be mutally orthogonal nonzero vectors in R^n. Then $\vec{F}_1, \vec{F}_2, \cdots \vec{F}_k$ are linearly independent

Ex

$$\begin{cases} \vec{F}_1 = (-4,0,0) \\ \vec{F}_2 = (0,-2,1) \\ \vec{F}_3 = (0,1,2) \end{cases} \qquad \begin{cases} \vec{F}_1 \cdot \vec{F}_2 = 0 \\ \vec{F}_1 \cdot \vec{F}_3 = 0 \\ \vec{F}_2 \cdot \vec{F}_3 = -2+2=0 \end{cases} \Rightarrow \textit{linearly independent.}$$

5. Basis

S is a subspace of R^n

$\vec{F}_1, \vec{F}_2, \cdots \vec{F}_k$ in S form a basis for S if

(1) $\vec{F}_1, \cdots \vec{F}_k$ are linearly independent.

(2)Every vector in S can be written as a linear combination of $\vec{F}_1, \cdots \vec{F}_k$

Ex

Cartesion unit vector $\vec{i}, \vec{j}, \vec{k}$ ***in*** R^n

Is $\vec{i}, \vec{j}, \vec{k}$ a basis for R^n?

Sol

$$\begin{cases} \vec{i} = (1,0,0) \\ \vec{j} = (0,1,0) \\ \vec{k} = (0,0,1) \end{cases}$$

$C_1\vec{i} + C_2\vec{j} + C_3\vec{k} = 0$ ， $C_1 = C_2 = C_3 = 0$

$\therefore \vec{i}, \vec{j}, \vec{k}$ *linearly independent*

$\Rightarrow \vec{i}, \vec{j}, \vec{k}$ form basis for R^3

any vector $\vec{F} = (x,y,z)$ in R^3

$$\vec{F} = (x,y,z) = (x,0,0) + (0,y,0) + (0,0,z)$$
$$= x(1,0,0) + y(0,1,0) + z(0,0,1)$$
$$x\vec{i} + y\vec{j} + z\vec{k}$$

6. Dimension of a subspace

the number of vectors in any basis of subspace of R^n.

Ex.

R^3 => Dimension 3

$\because$ Basis: $\vec{i}$, $\vec{j}$, $\vec{k}$

Ex

Rn => Dimension n

∵ Basis: $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$

Ex

S:{ $\vec{F}$ | $\vec{F}$: (x, y, z, 0, x-y, x+y, z) in R^6}

1. Is S a subspace of R^6
2. Basis of S
3. Dimension of S

Sol:

1. Subspace of R^6

A, x = y = z = 0 ∵ F = 0 ∈ S

B, $\vec{F}_1 = (x_1, y_1, 0, x_1 - y_1, x_1 + y_1, z_1)$

$\vec{F}_2 = (x_2, y_2, 0, x_2 - y_2, x_2 + y_2, z_2)$

$\vec{F}_1 + \vec{F}_2 = (x_1 + x_2, y_1 + y_2, 0, (x_1 + x_2) - (y_1 + y_2), (x_1 + x_2) + (y_1 + y_2), z_1 + z_2)$

$= (x, y, 0, x - y, x + y, z) \in S$

C, aF1 = (ax_1, ay1, 0, ax1−ay_1, ax1+ay_1, az1)

= (x, y, z, 0, x-y, x+y, z) ∈ S

⇨ S is subspace of R^6

2. Basis of S in R^6

(x, y, z, 0, x-y, x+y, z)

= (x,0,0,x,x,0) + (0,y,0,-y ,y+0) + (0,0,0,0,0,z)

= x(1,0,0,1,1,0) + y(0,1,0,-1,1,0) + z(0,0,0,0,0,1)

$= x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ form of a basis of S in R^6

3. Dimension of S, D = 3