Chap4 Series Solutions

4.1 Review

(1) Power series

$$\sum_{N=0}^{\infty} Cn(x-a)^n$$

a : center of series

Cn : coefficients

(2)Absolute convergence

Suppose $\sum_{N=0}^{\infty} Cn(x-a)^n$ converges for x = $x_1 \neq$ a

Then the series converges absolutely for all x such that $|x-a| < |x_1 - a|$

or $\sum_{N=0}^{\infty} \left|Cn(x-a)^n\right|$ converges

(3)Ratio test for convergence of series

$$\lim_{n\to\infty}\left|\frac{C_{n+1}(x-a)^{n+1}}{C_n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|(x-a) = L$$ if L<1 then the series converges

$\therefore \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|(x-a)$<1 or (x-a) < $\frac{1}{\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|(x-a)} = R$ Here R = radius of convergence

Ex: geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$ the series converges if

$$|x| < \frac{1}{\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|(x-a)} = 1 = R$$

Ex:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$ radius of convergence R

$$|x| < \frac{1}{\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right|} = \frac{1}{\lim_{n\to\infty}\left|\frac{1}{n+1}\right|} = \infty = R$$

Ex: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)9^n}(x-2)^{2n}$ Interval of convergence = ?

Sol. Using ratio test we have $\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+2)9^{n+1}}(x-2)^{2(n+1)}}{\frac{(-1)^n}{(n+1)9^n}(x-2)^{2n}}\right|$

$$=\lim_{n\to\infty}\left|\frac{-(n+1)}{(n+2)9}\right||x-2|^2$$

$$=\lim_{n\to\infty}\frac{n+1}{9(n+2)}|x-2|^2$$

$$=\frac{1}{9}|x-2|^2$$

The series converges if $\frac{1}{9}|x-2|^2 < 1$

or $|x-2|^2 < 9$

$|x-2| < 3$

or -3 < x-2 < 3

-1 < x < 5

Interval of converges if x < -1 or x > 5 series diverges

(4) Operations of power series

A. Add and Subtract

$$F(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

$$F(x)+g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$$

B. Multiplication

(a) $k \bullet F(x)$

$$k \bullet F(x) = F(x) = \sum_{n=0}^{\infty} ka_n (x - x_0)^n$$

$$k = cons\tan t$$

(b) $(Fg)(x) = F(x)g(x)$

$$F(x)g(x) = (\sum_{n=0}^{\infty} a_n (x - x_0)^n)(\sum_{n=0}^{\infty} b_n (x - x_0)^n) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

When $C_n \sum_{j=0}^{n} a_n b_{n-j}$

(c) Term by term differentiation

$$F(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{converges}$$

$$F'(x) = \sum_{n=0}^{\infty} na_n (x - x_0)^{n-1} \quad \text{converges}$$

$$F''(x) = \sum_{n=0}^{\infty} n(n-1)a_n (x - x_0)^{n-2}$$

$$\vdots$$

$$F^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)a_n (x - x_0)^{n-k}$$

(d) Term by term integration

$$F(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$\int F(x)dx = \sum_{n=0}^{\infty} \int a_n (x - x_0)^n dx + c$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{x+1} + c$$

(5) Shifting indices

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}$$

4.2 Power series solution through Recurrence Relations

1. Power Series method
   The power series method is the standard basic method for solving the linear D.E. with "variable" coefficients y'' + p(x) y' + q(x) y = r(x)
2. Existence of power series solutions if P(x) q(x) r(x) in D.E. y'' + p y' + q y = r are analytic at $x = x_0$, the every solution of above D.E. is analytic at $x = x_0$ and con be represented by a power series $\sum_{x=0}^{\infty} Cn(x - xo)^{n}$ with radius of convergence R>0.

3.Examples

(1) D.E. y' - 2xy = 0

$$\Rightarrow \quad \frac{dy}{dx} = 2xy$$

$$\ln y = x^2 + c$$

$$y = ke^{x^2}$$

Use power series method to solve the D.E.
Assume the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$\therefore y' - 2xy = \sum_{n=1}^{\infty} na_n x^{n-1} - 2x\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$= a_1 + \sum_{n=2}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$= a_1 + \sum_{k=1}^{\infty} a_{(k+1)} x^k - \sum_{k=1}^{\infty} 2a_{k-1} x^k$$

$$= a_1 + \sum_{k=1}^{\infty} [(k+1)a_{k+1} - 2a_{k-1}]x^k$$

$$= 0$$

for nontrivial solution $x^k \neq 0$

$\therefore a_1 = 0$

$(k+1)a_{k+1} - 2a_{k-1} = 0 \qquad k = 1,2,3,....$

$\therefore a_1 = 0, a_{k+1} = \dfrac{2}{k+1} a_{k-1} \qquad k = 1,2,3,....$

$k = 1 \qquad a_2 = a_0$

$k = 2 \qquad a_3 = \dfrac{2}{3} a_1 = 0$

$k = 3 \qquad a_4 = \dfrac{2}{4} a_2 = \dfrac{1}{2} a_0$

$k = 4 \qquad a_5 = \dfrac{2}{5} a_3 = 0$

$\therefore$ Solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \Lambda$$

$$= a_0 + a_2 x^2 + a_4 x^4 + \Lambda$$

$$= a_0 + a_0 x^2 + \frac{a_0}{2!} x^4 + \frac{a_0}{3!} x^6 + \Lambda$$

$$= a_0 \left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \Lambda \right]$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$= a_0 e^{x^2}$$

2. D.E. $y'' + xy' - y = e^{3x} \ldots (1)$

$p(x) = x, q(x) = -1, r(x) = e^{3x}$

$\because$ p(x), q(x), are analytic at $x=x_0$

$\therefore$ Power series solution can be expended about $x_0=0$

Assume solution $y(x) = \sum_{n=0}^{\infty} a_n x^n \ldots (2)$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n \, \mathrm{K} \; (3)$$

Sub (2) and (3) into (1) to obtain

$$\therefore \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(\frac{3^n}{n!} \right) x^n \, \Lambda \; (4)$$

Shift indices in 1st term of equation (4) to

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_n e^{2x^n}$$

$\therefore$Equation (4) become

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(\frac{3^n}{n\mathrm{I}} \right) x^n$$

Collecting term from n=1 on under one summation, we have

$$\sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n - a_n \right] x^n + 2a_2 - a_0 = 1 + \sum_{n=1}^{\infty} \left(\frac{3^n}{n!} \right) x^n \, \Lambda \; (5)$$

Equate coefficients on both side of equation (5) to yield

$$\begin{cases} 2a_2 - a_0 = 1 \\ (n+2)(n+1) a_{n+2} + (n-1) a_n = \dfrac{3^n}{n!} \end{cases}$$

$$\therefore \begin{cases} a_2 = \dfrac{1}{2}(1 + a_0) \\ a_{n+2} = \dfrac{\dfrac{3^n}{n!} + (1-n) a_n}{(n+2)(n+1)} \quad n = 1,2,3\Lambda \;\; \mathrm{Re}\,currence\,\mathrm{Re}\,lation \end{cases}$$

$$n = 1 \qquad a_3 = \frac{3}{(1+2)(1+1)} = \left(\frac{3}{3*2} \right)$$

$$n = 2 \qquad a_4 = \frac{\dfrac{9}{2} - a_2}{(2+2)(2+1)} = \frac{3^2}{2!4*3} - \frac{a_2}{4*3} = \frac{3^2}{4!} - \frac{2a_2}{4!} = \frac{3^2}{4!} - \frac{1+a_0}{4!}$$

$$n = 3 \qquad a_5 = \frac{\frac{3^3}{3!} - 2a_3}{(3+2)(3+1)} = \frac{3^3}{3!5*4} - \frac{2*3}{5!}$$

$$n = 4 \qquad a_6 = \frac{\frac{3^4}{4!} - 3a_4}{(4+2)(4+1)} = \frac{3^4}{6!} - \frac{3}{6*5}a_4 = \frac{3^4}{6!} - \frac{3}{6*5}\left(\frac{3^2}{4!} - \frac{1+a_0}{4!}\right)$$

$$= \frac{3^4 - 3^2}{6!} + \frac{3(1+a_0)}{6!}$$

M

$\therefore$ solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \Lambda$$

$$= a_0 + a_1 x + \frac{1}{2}(1+a_0)x^2 + \frac{1}{2}x^3 + \left(\frac{1}{3} - \frac{a_0}{24}\right)x^4 + \Lambda$$

4.3 Singular points and method of Frobenius

1. Ordinary and Singular points A point $x=x_0$ is an ordinary point of differential equation $P(x)y'' + Q(x)y' + R(x)y = F(x)$ or

$y'' + p(x)y' + q(x)y = f(x)$ if $p(x_0) \neq 0$ and $\frac{Q(x)}{P(x)}, \frac{R(x)}{P(x)} and \frac{F(x)}{P(x)}$ are analytic at x_0, that is p(x),q(x) and f(x) have a power series in $(x-x_0)$ with a positive radius of convergence. A point is not an ordinary point is a singular point.

EX : $y'' + e^x y' + (\sin x)y = 0$ Ordinary points? Singular points?

Sol : $\left.\begin{array}{l} p(x) = e^x \\ q(x) = \sin x \end{array}\right\}$ Continuous FN for every x

∴p(x), q(x) are analytic for every x every finite value of x is an ordinary point. Solution can be assumed as $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ where x_0 is any finite value.

EX : $y'' + (\ln x)y = 0$

Is x=0 an ordinary pt or singular pt?

$q(x) = \ln x$ Is not an analytic at x=0

∴x=0 is a singular pt.

EX : $xy'' + \sin xy = 0$

Is x=0 an ordinary pt or singular pt?

$$q(x) = \frac{\sin x}{x} = \lim_{x\to 0} q(x) = \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1(exists)$$

$$q(x) = \frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\Lambda$$

∵ $\frac{\sin x}{x}$ is analytic at x=0

∴ x=0 is an ordinary pt.

2. Regular and Irregular singular points

A singular points $x=x_0$ of D.E.

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$ ie

said to be a regular singular point if

$(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic

At X_0 otherwise a irregular singular point

Ex. $(x^2-4)^2 y'' + (x-2)y' + y = 0$

Singular point? Regular or Irregular?

Sol. $P(x) = (x^2-4)^2$

$$Q(x) = (x-2)$$

$$R(x) = 1$$

$$p(x) = \frac{Q(x)}{P(x)} = \frac{1}{(x-2)(x+2)^2}$$

$$q(x) = \frac{R(x)}{P(x)} = \frac{1}{(x^2-4)^2}$$

$\therefore x = 2$ & $x = -2$ are singular point

$\Theta\ p(x)$ & $q(x)$ are not analytic at $x = \pm 2$

Look at

$$(x-2)p(x) = \frac{1}{(x+2)^2}$$

$$(x-2)^2 q(x) = \frac{1}{(x+2)^2}$$ are analytic at $x = 2$

$\therefore x = 2$ is a regular singular point

Look at

$$(x-2)p(x) = \frac{1}{(x+2)(x-2)}$$

$$(x-2)^2 q(x) = \frac{1}{(x-2)^2}$$

$\Theta\ (x+2)p(x)$ is not analytic at $x = -2$

$\therefore x = -2$ is a irregular singular point

3. Existence of power series solution

If $x=x_0$ is an ordinary point of D.E.

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

Two linearly independent solutions in the form of power series

centered at x_0 , $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ can be founded

Ex. $y'' - 2xy = 0$

Sol.

$$\begin{cases} P(x) = 1 \\ Q(x) = 0 \\ R(x) = -2x \\ F(x) = 0 \end{cases}$$

X=0 is an ordinary point of D.E.

$y'' - 2xy' = 0$

$\therefore$ Solution y(x) can be assumed as

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\therefore y'' - 2xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2 a_n x^{n+1}$$

$$= 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2 a_n x^{n+1}$$

$$= 2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} 2 a_{k-1} x^k$$

$$= 2a_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) a_{k+2} - 2 a_{k-1} \right] x^k = 0$$

$\therefore 2a_2 = 0$

$(k+2)(k+1) a_{k+2} - 2a_{k-1} = 0$ k=1 , 2 , …….

Or $\left\{ \begin{array}{l} a_2 = 0 \\ a_{k+2} = \dfrac{2}{(k+2)(k+1)} a_{k-1}, k = 1,2,3..... \end{array} \right\}$ Recurrence relation

K=1 $a_3 = \frac{2}{3\times 2} a_0$

K=2 $a_4 = \frac{2}{4\times 3} a_1$

K=3 $a_5 = \frac{2}{5\times 4} a_2 = 0$

K=4 $a_6 = \frac{2}{6\times 5} a_3 = \frac{2^2}{6\times 5\times 3\times 2} a_0$

K=5 $a_7 = \frac{2}{7\times 6} a_4 = \frac{2^2}{7\times 6\times 4\times 3} a_1$

K=6 $a_8 = \frac{2}{8\times 7} a_5 = 0$

K=7 $a_9 = \frac{2}{9\times 8} a_6 = \frac{2^3}{9\times 8\times 6\times 5\times 3\times 2} a_0$

K=8 $a_{10} = \frac{2^3}{10\times 9\times 7\times 6\times 4\times 3} a_1$

K=9 $a_{11} = 0$

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_3 x^3 + a_4 x^4 + a_6 x^6 + ...$$

$$= a_0\left[1 + \frac{2}{3\times 2} x^3 + \frac{2^2}{6\times 5\times 3\times 2} x^6 + \frac{2^3}{9\times 8\times 6\times 5\times 3\times 2} x^9 + ...\right]$$

$$+ a_1\left[x + \frac{2}{4\times 3} x^4 + \frac{2^2}{7\times 6\times 4\times 3} x^7 + \frac{2^3}{10\times 9\times 7\times 6\times 4\times 3} x^{10} + ...\right]$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

Here

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{2^k\left[1\times 4\times 7\times ...(3k-2)\right]}{(3k)!} x^{3k}$$

$$y_2(x) = x + \sum_{k=1}^{\infty} \frac{2^k\left[2\times 5\times 8\times ...(3k-1)\right]}{(3k+1)!} x^{3k+1}$$

4. Method of Frobenius (solution about singular point)

If $x=x_0$ is a regular singular point of D.E.

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

then these exists at least one series solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

Where r is a constant to be determined (r= integer, negative integer or non-integer)

Ex. Solve $3xy'' + y' - y = 0$

Sol.

$$P(x) = 3x$$
$$Q(x) = 1$$
$$R(x) = -1$$

Let

$$P(x) = 3x = 0$$

$\Rightarrow x = 0$ is singular point

$\Theta\ (x - 0)Q(x) = x$

$(x-0)^2 R(x) = -x^2$ Analytic point

$\therefore x = 0$ is a regular singular point

According to method of Frobenius these exists at least one Frobenius solution about the regular point $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2}$$

$$\therefore 3xy'' + y' - y$$

$$= \sum_{n=0}^{\infty} 3(n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$= x^r \left[3r(r-1)C_0 x^{-1} + \sum_{n=1}^{\infty} 3(n+r)(n+r-1)C_n x^{n-1} + rC_0 x^{-1} + \sum_{n=1}^{\infty} (n+r)C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n \right]$$

$$= x^r \left\{ r(3r-2)C_0 x^{-1} + \sum_{n=0}^{\infty} \left[(k+r+1)(3k+3r+1)C_{k+1} - C_k \right] x^k \right\}$$

$$= 0$$

$$\therefore r(3r-2)C_0 = 0$$

$$(k+r+1)(3k+3r+1)C_{k+1} - C_k = 0$$ k=0,1,2……..

$$r(3r-2)=0 \qquad r_1 = \frac{2}{3}, r_2 = 0$$

$$r_1 = \frac{2}{3} \qquad \therefore C_{k+1} = \frac{C_k}{(3k+5)(k+1)}$$ k=0,1,2…..

K=0 $$C_1 = \frac{C_0}{5\times 1}$$

K=1 $$C_2 = \frac{C_1}{8\times 2} = \frac{C_0}{(5\times 1)(8\times 2)} = \frac{C_0}{2!5\times 8}$$

K=2 $$C_3 = \frac{C_2}{11\times 3} = \frac{C_0}{(5\times 1)(8\times 2)(11\times 3)} = \frac{C_0}{3!5\times 8\times 11}$$

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$$C_k = \frac{1}{k!5\times 8\times 11\times \ldots.(3k+2)} C_0$$ k=1, 2, 3…..

$$\therefore y_1 = \sum_{n=0}^{\infty} C_n x^{n+r_1} = \sum_{n=0}^{\infty} C_n x^{n+\frac{2}{3}}$$

$r_2=0$

$$\therefore C_{k+1} = \frac{C_k}{(k+1)(3k+1)}$$ k=0, 1, 2…

K=0 $$C_1 = \frac{C_0}{1\times 1}$$

K=1 $$C_2 = \frac{C_1}{2\times 4} = \frac{C_0}{(1\times 1)(2\times 4)} = \frac{C_0}{2!1\times 4}$$

K=2 $$C_3 = \frac{C_2}{3\times 7} = \frac{C_0}{(1\times 1)(2\times 4)(3\times 7)} = \frac{C_0}{3!1\times 4\times 7}$$

$$C_k = \frac{1}{k!1\times 4\times 7\times ...(3k-2)}C_0 \qquad \text{k=0, 1, 2….}$$

$$\therefore y_2(x) = \sum_{n=0}^{\infty} C_n x^{n+r_2} = \sum_{n=0}^{\infty} C_n x^n = C_0\left[1+\sum_{n=1}^{\infty}\frac{1}{n!1\times 4\times 7\times ..(3n-2)}x^n\right]$$

$\therefore$ General solution $\quad y(x) = C_1 y_1(x) + C_2 y_2(x)$

Ex. Solve $x^2 y'' + 5xy' + (x+4)y = 0$

$$y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r} + \sum_{n=0}^{\infty}5(n+r)C_n x^{n+r} + \sum_{n=0}^{\infty}C_n x^{n+r+1} + \sum_{n=0}^{\infty}4C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r} + \sum_{n=0}^{\infty}5(n+r)C_n x^{n+r} + \sum_{n=1}^{\infty}C_{n-1} x^{n+r} + \sum_{n=0}^{\infty}4C_n x^{n+r} = 0$$

$$[r(r-1)+5r+4]C_0 x^r + \sum_{n=1}^{\infty}[(n+r)(n+r-1)C_n + 5(n+r)C_n + C_{n-1} + 4C_n]x^{n+r} = 0$$

$$r(r-1)+5r+4=0$$

$$(n-2)(n-3)C_n + 5(n-2)C_n + C_{n-1} + 4C_n = 0$$

$$C_n = -\frac{1}{(n-2)(n-3)+5(n-2)+4}C_{n-1}$$

Or

$$C_n = -\frac{1}{n^2}C_{n-1} \qquad \text{for} \quad n=1,2,3\ldots$$

$$C_1 = -C_0$$

$$C_2 = -\frac{1}{4}C_1 = \frac{1}{4}C_0 = \frac{1}{(2)^2}C_0$$

$$C_3 = -\frac{1}{9}C_2 = -\frac{1}{4\times 9}C_0 = -\frac{1}{(2\times 3)^2}C_0$$

$$C_4 = -\frac{1}{16}C_3 = -\frac{1}{4\times 9\times 16}C_0 = \frac{1}{(2\times 3\times 4)^2}C_0$$

$$C_n = (-1)^n\frac{1}{(n!)^2}C_0$$

$$y(x) = C_0\left[x^{-2} - x^{-1} + \frac{1}{4} - \frac{1}{36}x + \frac{1}{576}x^2 + ...\right] = C_0\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}x^{n-2}$$

4.4 Second Solution and Logarithm Terms

1. Indicial Equation

If D.E. $y''+p(x)y'+q(x)y=0$ has regular singular point at $x=0$. Then multiplied by x^2, we have

$x^2y''+x^2p(x)y'+x^2q(x)y=0$ or

$x^2y''+xa(x)y'+b(x)y=0$

where $a(x)=xp(x)$, $b(x)=x^2q(x)$ are analytic at $x=0$. Therefore

$$\begin{cases} a(x)=xp(x)=p_0+p_1x+p_2x^2+\ldots\ldots \\ b(x)=x^2q(x)=q_0+q_1x+q_2x^2+\ldots\ldots \end{cases}$$

Assume Frobenius Solution

$$y(x)=\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$y'(x)=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}=x^{r-1}[rc_0+(r+1)c_1x+\ldots\ldots]$$

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}=x^{r-2}[r(r-1)c_0+(r+1)rc_1x+\ldots\ldots]$$

$$\therefore x^2y''+xa(x)y'+b(x)y$$

$$=x^r[r(r-1)c_0+(r+1)rc_1x+\ldots\ldots]+x^r[p_0+p_1x+p_2x^2+\ldots\ldots][rc_0+(r+1)c_1x+\ldots\ldots]$$

$$+x^r[q_0+q_1x+q_2x^2+\ldots\ldots][c_0+c_1x+\ldots\ldots]$$

$$=x^r[r(r-1)+rp_0+q_0]c_0+x^{r+1}\{c_1[r(r+1)+p_0(r+1)+q_0]+c_0[rp_1+q_1]\}$$

$$+x^{r+2}\{c_2[(r+1)(r+2)+p_0(r+2)+q_0]+c_1[p_1(r+1)+q_0]+c_0[rp_2+q_2]\}+\ldots\ldots$$

Let $f(r)=r(r-1)+p_0r+q_0$

$g_h(r)=p_n(r-n)+q_n$ n=1, 2 ……

$$\therefore\ x^2y''+xa(x)y'+b(x)y = f(r)c_0x^r + [c_1f(r+1)+c_0g_1(r+1)]x^{r+1}$$

$$+[f(r+1)c_2+g_1(r+2)c_1+g_2(r+2)c_0]x^{r+2}$$

$$\vdots$$

$$+[f(r+m)c_m+\sum_{n=1}^{m}g_n(r+m)c_{m-n}]\ x^{r+m}$$

$$\vdots$$

$$=0$$

Let $f(r)=0$, namely

$r(r-1)+p_0r+q_0=0$ Indicial equation

and all coefficient of each term are zero, then we have $(c_0 \neq 0)$

$$\begin{cases} f(r+1)c_1 = -g_1(r+1)c_0 \\ f(r+m)c_m = -\sum_{n=1}^{m}g_n(r+m)c_{m-n} \quad m \geqq 1 \end{cases}$$ Recurrence Formula

2. Cases of indicial roots

Suppose r_1 and r_2 are real roots and $r_1 > r_2$

(1) Case 1, distinct roots not differing by an integer, $r_1 \neq r_2$, $r_1 - r_2 \notin I$

Two linearly independent solutions exist

$$y_1 = \sum_{n=0}^{\infty}c_nx^{n+r_1} \qquad c_0 \neq 0$$

$$y_2 = \sum_{n=0}^{\infty}c_nx^{n+r_2} \qquad b_0 \neq 0$$

Ex. Solve $2xy''+(1+x)y'+y=0$

Sol.

$x=0$ is regular singular point

$\therefore$ Frobenius Solution is

$$y(x)=\sum_{n=0}^{\infty}c_nx^{n+r}$$

$\therefore\ 2xy''+(1+x)y'+y$

$$= x^r + \{r(2r-1)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k\}$$

$$= 0$$

Indicial equation

$$r(2r-1=0)$$

$$r_1 = \frac{1}{2}, \quad r_2 = 0$$

$$r_1 - r_2 = \frac{1}{2} \neq \text{Integer}$$

Recurrence formula

$$c_{k+1} \frac{k+r+1}{(k+r+1)(2k+2r+1)} c_k \qquad \text{k=0, 1, 2 ……}$$

$$r_1 = \frac{1}{2} \qquad c_{k+1} = \frac{k+\frac{3}{2}}{(k+\frac{3}{2})(2k+2)} c_k \qquad \text{k=0, 1, 2 ……}$$

$$\therefore y_1(x) = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!} x^{n+\frac{1}{2}} \qquad \text{converges for } x \geqq 0$$

$$r_2 = 0 \qquad c_{k+1} = \frac{k+1}{(k+1)(2k+1)} c_k \qquad \text{k=0, 1, 2 ……}$$

$$\therefore y_2(x) = c_0[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \times 3 \times 5 \times 7 \times \Lambda\ \Lambda \times (2n-1)} x^n] \quad \text{converges for } |x| < \infty$$

$\therefore$ on $(0, \infty)$

$$y(x) = c_1 y_1 + c_2 y_2$$

Case 2, $r_1 - r_2 = I$, I is a positive integer.

$$r = r_1 \quad y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

$$r = r_2 \quad y_2(x) = \sum_{n=0}^{\infty} c_n x^{n+r_2} = \sum_{n=0}^{\infty} c_n x^{n+r_1-I} = \sum_{n=I}^{\infty} c_{n-I} x^{n+r_1} \quad \in y_1(x)$$

$y_1(x)$ & $y_2(x)$ are linearly dependent

$\therefore$Only one solution $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ is obtained.

$y_1 = x^{r_1}(c_0 + c_1 x + c_2 x^2 + ...)$ is known, so the second solution $y_2(x)$ can be obtained by using "Reduction of Order"

$$\therefore \quad y_2(x) = u(x) y_1(x)$$

$$y_2'(x) = u' y_1 + u y_1'$$

$$y_2''(x) = u''y_1 + 2u'y_1' + uy_1''$$

Substituting above equations into the D.E.

$x^2 y''(x) + xa(x)y'(x) + b(x)y(x) = 0$, we have

$$x^2(u''y_1 + 2u'y' + uy_1'') + xa(x)(u'y_1 + uy_1') + b(x)u(y_1)$$
$$= x^2 y_1 u'' + 2x^2 y_1' u' + xa(x)y_1 u'$$
$$= 0$$

Divided by $x^2 y$ yield

$$u'' + [2\frac{y_1'}{y_1} + \frac{a(x)}{x}]\ u' = 0$$

$$a(x) = p_0 + p_1 x + p_2 x^2 + ...$$

$$\frac{y_1'}{y_1} = \frac{\sum_{n=0}^{\infty}(n+r_1)c_n x^{n+r_1-1}}{\sum_{n=0}^{\infty} c_n x^{n+r_1}} = \frac{x^{r_1-1}[rc_0 + (r_1+1)c_1 x + ...]}{x^{r_1}[c_0 + c_1 x + ...]} = \frac{r_1}{x} + ...$$

$$\therefore u'' + [\frac{2r_1}{x} + ... + \frac{p_0}{x} + p_1 + ...]\ u' = u'' + [\frac{2r_1 + p_0}{x} + ...]\ u' = 0$$

From indicial equation, we have

$$r(r+1)p_0 r + q_0 = 0$$

$$\therefore \quad r^2 + (p_0 - 1)r + q_0 = 0$$

$$\begin{cases} r_1 + r_2 = -(p_0 - 1) \\ r_1 r_2 = q_0 \end{cases}$$

$$\because \quad r_1 - r_2 = I \qquad (r_2 = r_1 - I)$$

$$\therefore \quad 1 - p_0 = r_1 + r_2 = r_1 + r_1 - I = 2r_1 - I$$

$$\therefore \quad 2r_1 + p_0 = I + 1$$

$$\therefore \quad u'' + [\frac{I+1}{x} + ...]\ u' = 0 \quad \text{or}$$

$$\frac{u''}{u'} = -[\frac{I+1}{x} + ...]$$

Integrating above equation yields $\ln u' = -(I+1)\ln x + \ln e^{(......)}$

$$= \ln \frac{e^{(O(x))}}{x^{N+1}} \qquad (r_1 - r_2 = I = N)$$

and $e^{(O(x))} = \sum_{n=0}^{\infty} k_n x^n$

$\therefore \quad u' = \frac{1}{x^{N+1}}(k_0 + k_1 + k_2 x^2 + ...)$

$$u(x) = (-\frac{k_0}{Nx^N} - ...... + k_N \ln x + k_{N+1} + ...)$$

$\therefore$ Second solution

$$y_2(x) = u(x) y_1(x)$$

$$= (-\frac{k_0}{Nx^N} - ...... + k_N \ln x + k_{N+1} x + ...) x^{r_1}(c_0 + c_1 x + c_2 x^2 + ...)$$

$$= (-\frac{k_0}{Nx^N} - ...... + k_{N+1} x + ...) x^{r_1}(c_0 + c_1 x + c_2 x^2 + ...) + k_N (\ln x) y_1(x)$$

$$= x^{-N}(-\frac{k_0}{N} - ...... - k_{N-1} x^N + k_{N+1} x^{N+1} + ...) x^{r_1}(c_0 + c_1 x + ...) + k_N (\ln x) y_1(x)$$

$$= x^{r_1 - N}(b_0 + b_1 x + b_2 x^2 + ...) + k_N (\ln x) y_1(x)$$

$$= k_N (\ln x) y_1(x) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad \text{or}$$

$$y_2(x) = k_N (\ln x) y_1(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Then, substitute $y_2(x)$ into the D.E. and obtain an equation for k_n

and a recurrence relation for b_n.

If $k_n = 0$, a second Frobenius solution $y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$ is obtained;

If $k_n \neq 0$, then second solution has a logarithm terms.

Ex. Solve $x^2 y'' + x^2 y' - 2y = 0$

Sol.

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} = x^{r-1}[r c_0 + (r+1) c_1 x + ...]$$

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}=x^{r-2}[r(r-1)c_0+(r+1)rc_1x+......]$$

$$\therefore x^2y''+x^2y'-2y$$

$$=[r(r-1)-2]c_0x^r+\sum_{n=1}^{\infty}[(n+r)(n+r-1)c_n+(n+r-1)c_{n-1}-2c_n]x^{n+r}$$

$$=0$$

Indicial equation

$$r(r-1)-2=0$$

$r_1=2$, $r_2=-1$

$r_1-r_2=3$ = Integer

Recurrence formula

$(n+r)(n+r-1)c_n+(n+r-1)c_{n-1}-2c_n=0$ n=0, 1, 2 ……

$r_1=2$ $c_n=-\dfrac{n+1}{n(n+3)}c_{n-1}$ n=0, 1, 2 ……

$$\therefore y_1(x)=c_0x^2[1-\frac{1}{2}x+\frac{3}{20}x^2-\frac{1}{30}x^3+\frac{1}{168}x^4-\frac{1}{1120}x^5+\frac{1}{8640}x^6+......]$$

$r_2=-1$ $(n-1)(n-2)c_n^*+(n-2)c_{n-1}^*-2c_n^*=0$ n=0, 1, 2 ……

$$\therefore y_2(x)=c_0^*\frac{1}{x}+c_1^*$$

$$\therefore x^2(2c_0^*x^{-3})+x^2(-c_0^*x^{-2})-2(c_1^*+c_0^*\frac{1}{x})=-c_0^*-2c_1^*=0$$

$$\therefore y_2(x)=c_0^*(\frac{1}{x}-\frac{1}{2})$$

EX 4.15

D.E. $xy'' - y = 0$

Sol: x=0 is regular singular point Assume Frobenius solution y(x)

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substitute y(x) into D.E. and shift indices to have

$$(r^2 - r)c_0 x^{r-1} + \sum_{n=1}^{\infty} [(n+r)(n+r-1)c_n - c_{n+1}]x^{n+r-1} = 0 \quad c_0=0$$

Indicial equation r^2-r=0

Recurrence relation

$$c_n = \frac{1}{(n+r)(n+r-1)} c_{n-1} \quad \text{n=1, 2...}$$

Two roots for indicial equation are

r_1=1 r_2=0 and $r_1 - r_2 = 1 \in N^+$

For r=r_1=1 we have

$$c_n = \frac{c_{n-1}}{n(n+1)} \quad \text{n=1, 2...}$$

After tedious calculation, we finally have

$$Cn = \frac{1}{n!(n+1)!} C_0 \quad \text{n=1, 2…}$$

$\therefore$ The first Frobenius solution is

$$y_1(x) = x^{r1} \sum_{n=0}^{\infty} C_n x^n$$

$$= x \sum_{n=0}^{\infty} C_0 \frac{x^n}{n!(n+1)!}$$

$$= C_0 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1}$$

For r = r_2 = 0 we have $n(n-1)C_n = C_{n-1}$ n=1, 2… n=1 0=C_0 Contrary to the assumption that $C_0 \neq 0$ $\therefore$ we can not find the second Frobenius solution by simply putting r=r2=0 into the recurrence relation.

Try a second solution

$$y_2(x) = k(\ln x)y_1(x) + \sum_{n=0}^{\infty} b_n x^{n+r2}$$

$$= k(\ln x)y_1(x) + \sum_{n=0}^{\infty} b_n x^n \,(r_2 = 0)$$

$$y_2{}'(x) = k(\ln x)y_1{}'(x) + \frac{k}{x}y_1 \sum_{n=0}^{\infty} b_n n x^{n-1}$$

$$y_2{}''(x) = k(\ln x)y_1{}''(x) + \frac{2k}{x}y_1{}' - \frac{k}{x^2}y_1 + \sum_{n=0}^{\infty} b_n n(n-1)x^{n-2}$$

$$\therefore xy_2{}'' - y_2$$

$$= x[k(\ln x)y_1{}'' + \frac{2k}{x}y_1{}' - \frac{k}{x^2}y_1 + \sum_{n=0}^{\infty} b_n n(n-1)x^{n-2}] - k(\ln x)y_1(x) - \sum_{n=0}^{\infty} b_b x^n = 0$$

$$k(\ln x)\left[xy_1'' - y_1\right] = 0$$

$$\therefore 2ky_1' - \frac{k}{x}y_1 + \sum_{n=0}^{\infty} b_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} b_n x^n = 0$$

Substitute $y_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n-1}$ (C_0 =1) into above equation and shift indices to obtains

$$(k - b_0)x^0 + \sum_{n=1}^{\infty} \left[\frac{2k}{(n!)^2} - \frac{k}{n!(n+1)!} + n(n+1)b_{n+1} - b_n \right] x^u = 0$$

$\therefore$ Equation for k is k-b_0=0

Recurrence relation for b_n is

$$\frac{2k}{(n!)^2} - \frac{k}{n!(n+1)!} + n(n+1)b_{n+1} - b_n = 0 \qquad \text{n=1, 2, 3……}$$

Thus, k=b_0 and recurrence relation is new are

$$b_{n+1} = \frac{1}{n(n+1)}\left[b_n - \frac{(2n+1)k}{n!(n+1)!} \right] \qquad \text{n=1, 2, 3…}$$

Choose b_0 =1 then k=1

$$\therefore b_{n+1} = \frac{1}{n(n+1)}\left[b_n - \frac{(2n+1)}{n!(n+1)!} \right] \qquad \text{n=1, 2, 3……}$$

N=1 $b_2 = \frac{1}{2}\left[b_1 - \frac{3}{2!} \right]$

N=2 $b_3 = \frac{1}{6}\left[b_2 - \frac{5}{2!3!} \right]$

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For a particular second solution, let $b_1 = 0$, obtaining

$$y_2(x) = k(\ln x)y_1 + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad (k=1\ b_1=0)$$

$$= (\ln x)y_1 + 1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 \ldots\ldots$$

(3) Case 3 Double roots $r_1 = r_2 = \dfrac{1-p_0}{2}$

First solution is

$$y_1(x) = x^{r_1}(c_0 + c_1 x + c_2 x^2 + \ldots\ldots)$$

Let second solution be

$$y_2(x) = u(x)y_1(x)$$

$\therefore$ D.E. $x^2 y'' + xa(x)y' + b(x)y = 0$

Reduces to

$$u'' + \left(\frac{2y_1'}{y_1} + \frac{p_0}{x} + p_1 + \ldots\right)u' = 0$$

$$\frac{y_1'}{y_1} = \frac{r_1}{x} + o(1)$$

$$\therefore u'' + \left(\frac{2r_1 + p_0}{x} + o(1)\right)u' = 0$$

$$\Theta\ r_1 = \frac{1-p_0}{2}$$

$$\therefore 2r_1 + p_0 = 1$$

$$\therefore u'' + \left(\frac{1}{x} + o(1)\right)u' = 0$$

$$\frac{u''}{u'} = -\left(\frac{1}{x} + o(1)\right)$$

By integration, we have

$$\ln u' = -\ln x + \ln e^{o(x)}$$

$$= -\ln x + \ln \sum_{n=0}^{\infty} k_n x^n$$

$$= \ln\left[\frac{1}{x}\sum_{n=0}^{\infty} k_n x^n\right]$$

$$\therefore u' = \frac{1}{x}\sum_{n=0}^{\infty} k_n x^n$$

$$= \frac{1}{x}\left(k_0 + k_1 x + k_2 x^2 + ...\right)$$

$$u(x) = k_{01}\ln x + k_1' x + k_2' x^2 + ...$$

$\therefore$ second solution $y_2(x)$ is

$$y_2(x) = u(x) y_1(x)$$

$$= k_0(\ln x) y_1(x) + (k_1' x + k_2' + \Lambda)\, y_1(x)$$

$$= k_0(\ln x) y_1(x) + \left(k_1' x + k_2' x^2 + \Lambda\right) x^{r_1}\left(c_0 + c_1 x + c_2 x^2 + \Lambda\right)$$

$$= k_0(\ln x) y_1(x) + \left(b_1 x + b_2 x^2 + \Lambda\right) x^{r_1}$$

$$\therefore y^2(x) = k_0(\ln x) y_1(x) + \sum_{n=1}^{\infty} b_n x^{n+r_1}$$

EX 4.11 and 4.13

$$x^2 y'' + 5xy' + (x+4)y = 0$$

Sol: x=0 regular singular point. Frobenius solution is $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$

Sub y(x) into D.E. and shift indices to yield

$$[r(r-1) + 5r + 4]c_0 x^r + \sum_{n=1}^{\infty}\left[(n+r)(n+r-1)c_n + 5(n+r)c_n + c_{n-1} + 4c_n\right]x^{n+r} = 0$$

Indicial equation $r^2 + 4r + 4 = (r+2)^2 = 0$ r_1=r_2=2 (Double roots)

Recurrence solution

$$c_n = \frac{-c_{n-1}}{(n+1)(n+r-1) + 5(n+r) + 4} \quad \text{n=1, 2...}$$

For r_1=-2 we have $c_n = -\frac{1}{n^2} c_{n-1}$ n=1, 2, 3...

After tedious calculation we finally have

$$c_n = (-1)^n \frac{1}{(n!)^2} c_0 \quad \text{n=1, 2...}$$

$\therefore$ The 1st solution $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^{n-2}$$

$\because r_1=r_2=-2$

$\therefore$ A second solution is of the form $y_2(x) = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}$ $\quad (k_0=1)$

Substitute $y_2(x)$ and $y_1(x)$ into the D.E. we finally have

$$(b_1-2)x^{-1} + \sum_{n=2}^{\infty}\left[\frac{4(-1)^n}{(n!)^2} + \frac{2(-1)^n}{(n!)^2}(n-2) + (n-2)(n-3)b_n + 5(n-2)b_n + b_{n-1} + 4b_n\right]x^{n-2} = 0$$

$$\therefore \begin{cases} b_1 - 2 = 0 \\ b_n = \dfrac{-1}{n^2}b_{n-1} - \dfrac{2(-1)^n}{n(n!)^2} \end{cases} \qquad n=2, 3, 4...$$

n=1 $\qquad b_1=2$

n=2 $\qquad b_2 = -\dfrac{1}{2^2}b_1 - \dfrac{2}{2(2!)^2} = -\dfrac{3}{4}$

n=3 $\qquad b_3 = -\dfrac{1}{3^2}b_2 + \dfrac{2}{3(3!)^2} = \dfrac{11}{108}$

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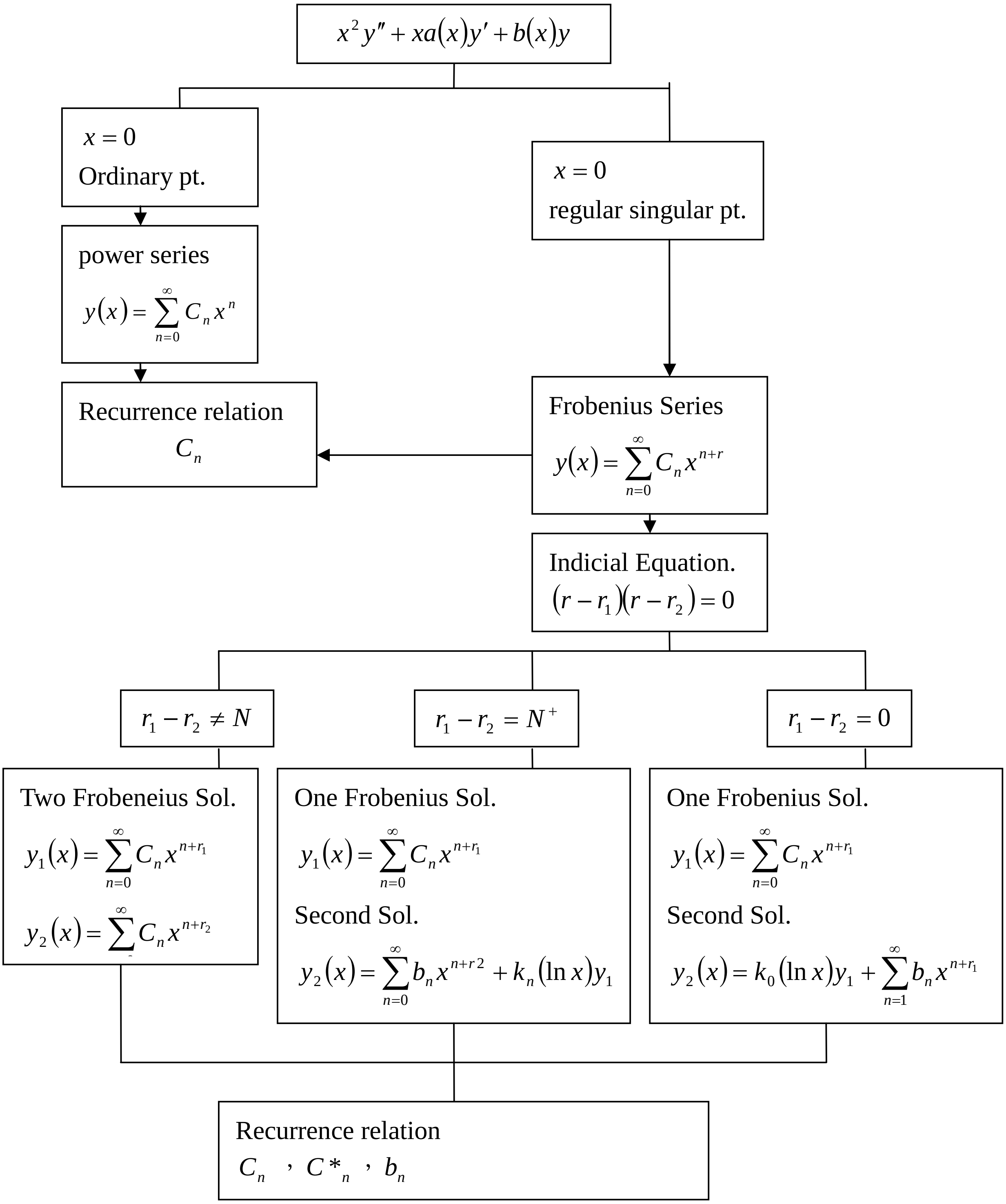
$\therefore$ Second solution $y_2(x)$ is

$$y_2(x) = (\ln x)y_1 + \sum_{n=1}^{\infty} b_n x^{n-2} = (\ln x)y_1 + \frac{2}{x} - \frac{3}{4} + \frac{11}{108}x + 0(x^2)$$

$\because$ y1(x) and y2(x) linearly independent

$\therefore$ general sol is $y(x) = c_1 y_1(x) + c_2 y_2(x)$


$x^2 y'' + xa(x)y' + b(x)y$
$x = 0$
Ordinary pt.
power series
$y(x) = \sum_{n=0}^{\infty} C_n x^n$
Recurrence relation
C_n
$x = 0$
regular singular pt.
Frobenius Series
$y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$
Indicial Equation.
$(r - r_1)(r - r_2) = 0$
$r_1 - r_2 \neq N$
$r_1 - r_2 = N^+$
$r_1 - r_2 = 0$
Two Frobeneius Sol.
$y_1(x) = \sum_{n=0}^{\infty} C_n x^{n+r_1}$
$y_2(x) = \sum^{\infty} C_n x^{n+r_2}$
One Frobenius Sol.
$y_1(x) = \sum_{n=0}^{\infty} C_n x^{n+r_1}$
Second Sol.
$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r2} + k_n (\ln x) y_1$
One Frobenius Sol.
$y_1(x) = \sum_{n=0}^{\infty} C_n x^{n+r_1}$
Second Sol.
$y_2(x) = k_0 (\ln x) y_1 + \sum_{n=1}^{\infty} b_n x^{n+r_1}$
Recurrence relation
C_n ， $C*_n$ ， b_n