Chap. 2 Second Order D.E.

2-1 Preliminary Concepts

1. 2^{nd}-order D.E.

   $f(x, y, y', y'') = 0$

2. solution of $f(x, y, y', y'') = 0$

   $f(x, \varphi(x), \varphi'(x), \varphi''(x)) = 0$ for all x in interval I.

   $\varphi(x)$ is a solution of linear 2^{nd}-order D.E.

3. linear 2^{nd}-order D.E.

   $y'' + p(x)\, y' + q(x)\, y = F(x)$

   $p(x)$, $q(x)$, $f(x)$, continuous

2-2 Theory Of Solution Of y''+ p(x) y'+ q(x) y=f(x)

1. initial-value problem

(1) initial valued problem

$y'' + p(x)y' + q(x)y = F(x)$

initial conditions

$y(x_0) = A$, $y'(x_0) = B$, A, B are arbitrary constants

(2) existence of a unique solution

Let p(x), q(x) and f(x) be continuous on an open interval I.

IF $x = x_0$ is any point in I, then a solution y(x) of the I.V.P exists on the interval I and is unique.

Ex. I.V.P $y'' + (\cos x)y' + e^x y = x^2$

$y(1) = 3$, $y'(1) = -5$

unique solution defined for all x $(-\infty < x < \infty)$

$\because \cos(x)$, e^x, x^2 are continuous on $-\infty < x < \infty$

Ex I.V.P $y'' + xy' + y = \tan(x)$

$y(0) = 1$ $y'(0) = -2$

unique solution exists on $-\pi/2 < x < \pi/2$

$\because \tan(x)$ continuous on $-\pi/2 < x < \pi/2$

2. linear dependence and linear independence

(1) linear dependence 線性相依

Two functions f(x) and g(x) are linearly dependent on an interval I, if there exists constant c1 and c2，not all zero，such that

$c_1 f(x) + c_2 g(x) = 0$

or $f(x) = c g(x)$ for all x in I.

$$f(x) = -\frac{C1}{C2} g(x) = cg(x)$$

ex. $f(x) = \sin 2x$ $I = (-\infty, \infty)$

$g(x) = \sin x \cos x$

f(x)、g(x) linear dependence?

(sol)

$c_1 \sin 2x + c_2 \sin x \cos x = 0$

$2c_1 \sin x \cos x + c_2 \sin x \cos 2x = 0$

$2c_1 + c_2 = 0$

choose $c_1 = 1$ $c_2 = -2$ $\neq 0$

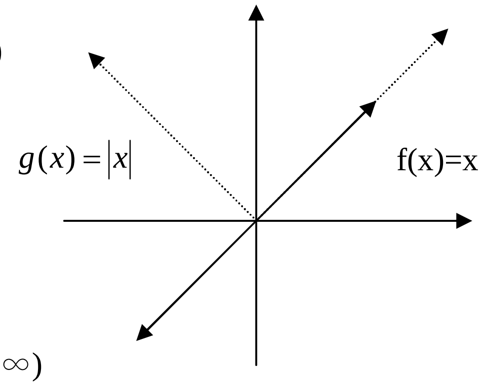
$\therefore \sin 2x$，$\sin x \cos 2x$ linear dependence or $\dfrac{f(x)}{g(x)} = \dfrac{\sin 2x}{\sin x \cos 2x} = 2$

f(x) =2g(x)

(2) linear independence　線性獨立

Two functions f(x) and g(x) are linearly independence on I if the only constant for which c1f(x) +c2g(x) =0 are c1=c2=0, or if neither one is constant multiple of the other for all x in I.

ex.　f(x)= x,　　$g(x) = |x|$　　I=(-∞,∞)



c1f+c2g=0

c1=c2=0

∴f(x)＆g(x) linear independence on (-∞,∞)

if　0<x<∞, then g(x)= x =f(x)

∴f(x)＆g(x) linear dependence on (0,∞)

(3) Wronskian

Assume that y1(x) and y2(x) possess at lease first derivatives on interval I. If the determinant (or Wronskian)

$$W(y1(x),y2(x)) = W = \begin{vmatrix} y1(x) & y2(x) \\ y1'(x) & y2'(x) \end{vmatrix} \neq 0$$

for at lease one point in interval I, then the function y1(x) and y2(x) are linearly independent on I.　On the other hand, if y1(x) and y2(x) are linearly dependent, then *W=0. (Note: W=0 is a necessary not sufficient condition for linear dependence of functions.)*

ex.　y1(x) $= e^{2x}$,　　　y2(x) $= e^{-2x}$

linear dependence or independence?

$$W(y1,y2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2-2 = -4 \neq 0 \text{ for all x}$$

∴y1&y2 linearly independent on (-∞,∞)

3. Solutions of linear D.E

(1) homogeneous equation

y”+p(x) y’+q(x) y=0

A. Theorem of superposition principle

Let y1(x) and y2(x) be solutions of y”+p(x) y’+q(x) y=0 on interval I. Then the linear combination y=c1y1+c2y2 where c1 and c2 are arbitrary constants, is also a solution on I.

pf $(c_1y_1+c_2y_2)''+p(x)(c_1y_1+c_2y_2)'+q(x)(c_1y_1+c_2y_2)$

$=c_1(y_1''+p(x)y_1'+q(x)y_1)+c_2(y_2''+p(x)y_2'+q(x)y_2)$

(each parenthesis $= 0$)

$=0$

B. Theorem of Wronskian test

Let y_1 and y_2 be solutions of $y''+p(x)y'+q(x)y=0$ on I. p(x) and q(x) are continuous on I. Then y_1 and y_2 are linearly independent on I iff $W(y_1, y_2)\neq 0$

for every x in I.

Ex. D.E $y''+y'=0$, solutions $y_1=\sin x$, $y_2=\cos x$

y_1&y_2 linearly independent?

Sol.

W(sinx, cosx)= $\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$ for all x

=> y_1, y_2 are linearly independent solutions

C. Fundamental set of solutions (基本組解)

Two solutions of $y''+p(x)y'+q(x)y=0$ are said to form a fundamental set of solutions on I if they are linearly independent on I.

D. General solution (通解)

Let y_1 and y_2 be a fundamental set of solutions of $y''+p(x)y'+q(x)y=0$ on I. The linear combination $c_1y_1+c_2y_2$ are the general solutions of the D.E on I.

Ex . $y''+y'=0$ solutions

$y_1=\sin x$, $y_2=\cos x$

$c_1y_1+c_2y_2$ general solution?

Sol W(sinx, cosx)= $\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$ for all x

⇨ y_1, y_2 are linearly independent solutions

⇨ y_1, y_2 fundamental set of solutions of $y'' + y = 0$

⇨ $c_1y_1+c_2y_2$ is a general solution of $y'' + y = 0$

Ex. $y'' + y = 0$ solutions

$y_1=\sin x$, $y_2=2\sin x$

$c_1y_1+c_2y_2$ general solution?

Sol $y_2 = 2\sin x = 2y_1$

⇨ y_1, y_2 are linearly dependent

⇨ $c_1y_1+c_2y_2 = c_1y_1+2c_2y_1 = (c_1+c_2)y_1 = cy_1$

└> not a general solution

(2) Nonhomogeneous equation

$y''+p(x)\, y'+q(x)\, y=f(x) \neq 0$

A. Theorem

Let y_1 and y_2 be linearly independent solutions of $y''+p(x)\, y'+q(x)\, y=0$ on I and let y_p be any particular solution of $y''+p(x)\, y'+q(x)\, y=f(x)$ on I.

Then $\varphi= c_1y_1+c_2y_2+ y_p$ is also a solution of $y''+p(x)\, y'+q(x)\, y=f(x)$ on I for any constants C_1 and C_2.

Pf.

$\varphi''+ p(x)\, \varphi'+q(x)\, \varphi$

$= C_1\, (y_1''+ py_1'+q\, y_1) +C_2\, (y_2''+ py_1'+q\, y_1) + y_p''+py_p'+qy_p$

$= y_p''+py_p'+qy_p =f(x)$

B. General solution

$y=c_1y_1+c_2y_2+ y_p(x) = y_c(x)+ y_p(x)$ is a general solution of $y''+p(x)y'+q(x)y=f(x)$ if $y_p(x)$ is a given solution of $y''+p(x)y'+q(x)y=f(x)$ on I and $y_c = c_1y_1+c_2y_2$ denotes the general solution of $y''+py'+qy=0$.

Ex. $y''+ y=x$ general solution ?

$y_p(x) =x$ is solution of $y''+ y=x$

$y_c(x)= c_1\cos x+c_2\sin x$ is general solution of $y''+ y=0$

general solution of given D.E. is $y(x)= y_c(x)+ y_p(x)$

2.3 Reduction of order.

Homogeneous linear 2[nd] -order D.E.

$y''+p(x)y'+q(x)y=0$, $p(x),q(x)$ continuous on I……………………….(1)

$y_1(x)$ is a known solution on I and $y_1(x)$ doesn't equal to 0 for all x in I .

How to find the second solution?

$y_2(x)$ is linearly independent to $y_1(x)$.

General solution:

$C_1y_1+C_2y_2$,

Method of solution:

Let $y_2(x)=u(x)y_1(x)$,

$y_2'(x)=uy_1'+u'y_1$,

$y_2''=uy_1''+2u'y'+y_1u''$,
$y_2''+p(x)y_2'+q(x)y_2=u[y_1''+p(x)y'+q(x)y_1]+y_1u''+(2y_1'+py_1)u'=0$,

Henceforth

$y_1u''+(2y_1'+py_1)u'=0$,

or $u''+[(2y_1'+py_1)/y_1]u'=0$,

Let $v(x)=u'(x)$, $g(x)=(2y_1'+py_1)/y_1$,

so $v'+g(x)v=0$,

$dv/v=-g(x)dx$,

$\int g(x)dx=\int (2y_1'+py_1)/y_1\, dx =2\int y_1'/y_1\, dx+\int p(x)dx=2\ln y_1+\int p(x)dx$.

So that

$v(x) = C1e^{-\int g(x)dx)}$

$=C1e^{(-2lny1-\int p(x)dx)}$

$=(C_1e^{-\int p(x)dx})/y_1^2$,

$u'(x)=v$,

so $u(x) = \int v(x)dx+C_2$

$=C_1 \int (e^{-\int pdx})/y_1^2dx+C_2$,

so $y_2(x)=u(x)y_1=C_1y_1(x) \int (e^{-\int pdx})/y_1^2dx+C_2y_1(x)$,

Choose $C_1=1$, $C_2=0$, then 2nd solution $y_2(x)$ is obtained.

$y_2(x)=y_1(x) \int (e^{-\int p(x)dx})/y_1^2dx$

Check if y_1& y_2 are linearly independent ?

$W(y_{1,} y_2)=e^{-\int p(x)dx} \neq 0$ for all x ,

y_1 and y_2 are linearly independent solutions.

$C_1y_1+C_2y_2$ is a general solution of $y''+py'+qy=0$.

Ex:

Given: D.E. $y''+4y'+4y=0$, $y_1=e^{-2x}$.

Find: general solution.

Solution:

Use the method of reduction of order to obtain the 2nd solution.

Let $y_2(x) = u(x)$ and $y_1 = u(x)e^{-2x}$,

$y_2' = -2u\, e^{-2x}+u' e^{-2x} = (u'-2u)\, e^{-2x}$

$y_2'' = -2(u'-2u)\, e^{-2x}+(u''-2u')\, e^{-2x}=(u''-4u'+4u)\, e^{-2x}$

$y_2''+4y_2'+4y_2=[(u''-4u'+4u)+4(u'-2u)+4u]e^{-2x}=0$,

So $u''\, e^{-2x}=0$,

$u''=0$,

$u(x)=C_1x+C_2$

Choose $C_1=1$, $C_2=0$,

$u(x)=x$,$y_2=u(x)y_1=xe^{-2x}$

Check if $W(y_2,y_1)\neq 0$,

General solution :

$y(x)=C_1y_1+C_2y_2=C\, e^{-2x}+C_2x\, e^{-2x}=(C_1+ C_2x)\, e^{-2x}$

2.4 Constant Coefficients Homogeneous Linear Equations

Homogeneous 2nd order linear D.E. with constant coefficients

$y'' + Ay' + By = 0$, A and B are real numbers.

Let $y=e^{\lambda x}$, $y'= \lambda e^{\lambda x}$, $y''= \lambda^2 e^{\lambda x}$.

So that

$(\lambda^2+A\lambda+B)e^{\lambda x} = 0$ ($e^{\lambda x} \neq 0$ for real x)

Hence,

$\lambda^2+A\lambda+B=0$ ………… Characteristic (Auxiliary) equation

1. cass1: Real distance roots , $A^2-4B>0$

   $\lambda_1=(-A\pm\sqrt{(A^2-4B)})/2$

   two solutions : $y_1=e^{\lambda_1 x}$, $y_2= e^{\lambda_2 x}$

   $W(y_1, y_2)=(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)x} \neq 0$

   So, y_1, y_2 linearly independent solutions .

   Hence general solutions:

   $y(x) = C_1 e^{\lambda_1 x}+C_2 e^{\lambda_2 x}$

   $= e^{\alpha x}[C_1\cosh(\beta x)+C_2\sinh(\beta x)]$

   where α, β depend on A and B ;

   $\cosh x=(e^x+e^{-x})/2$, $\sinh x=(e^x-e^{-x})/2$.

2. cass2: Repeated roots : $A^2-4B=0$

roots : $\lambda_1=\lambda_2=-A/2$,

Only one solution $y_1= e^{\lambda 1x}=e^{-A/2x}$ is obtained

By using the method "Reduction of Order", the second solution can be obtained.

Let $y_2(x)=u(x)y_1(x)$,

$y_2(x)= y_1(x) \int (e^{\int -pdx})/y_1^2(x)dx$, $(p(x)=A)$

$=e^{\lambda 1x} \int [(e^{-Ax})/(e^{-A/2x})^2]dx= e^{\lambda 1x} \int [e^{-Ax}/ e^{-Ax}]dx=x\, e^{\lambda 1x}$.

$e^{\lambda 1x}$, $x\, e^{\lambda 1x}$ are linearly independent,

general solution is $y(x)=C_1 e^{\lambda 1x}+C_2 x\, e^{\lambda 1x}$.

3. case3: Complex conjugate roots : $A^2-4B<0$,

roots: $\lambda_1=(-A+i\sqrt{(4B-A^2)})/2=p+iq$, $\lambda_2=(-A-i\sqrt{(4B-A^2)})/2=p-iq$

two solutions : $y_1(x)=e^{(p+iq)x}$, $y_2(x)=e^{(p-iq)x}$

From Euler formula, we have

$e^{\pm i\theta}=\cos\theta \pm i\sin\theta$.

So

$y_1(x)=e^{(p+iq)x}=e^{px}e^{iqx}=e^{px}(\cos(qx)+i\sin(qx))$,
$y_2(x)=e^{(p-iq)x}=e^{px}e^{-iqx}=e^{px}(\cos(qx)-i\sin(qx))$.

General solution

$y(x)=C_1y_1+C_2y_2=e^{px}[(C_1+C_2)\cos(qx)+i(C_1-C_2)\sin(qx)]$

choose $C_1=C_2=1/2$ to obtain one solution .

$y_3(x)=e^{px}\cos qx$

choose $C_1=1/2i$, $C_2=-1/2i$ to obtain other solution .

$y_4(x)=e^{px}\sin qx$

Use $W(y_3,y_4)$ to check if y_3 and y_4 are linearly independent solution .

$W(y_3,y_4)=e^{2px} \neq 0$ for all real x ,

so $e^{px}\cos qx$, $e^{px}\sin qx$ are linearly independent .

general solution is

$y(x)=C_1e^{px}\cos(qx)+C_2e^{px}\sin(qx)$

2.5 Euler Equations

1. Definition

A 2nd-order homogeneous equation

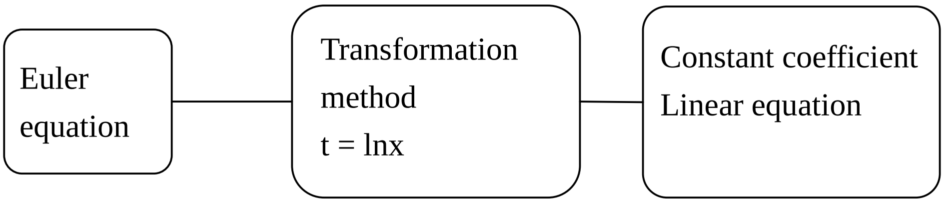
$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0 \qquad \text{a,b,c, constants}$$

or $$y''+\frac{A}{x}y'+\frac{B}{x^2}y=0 \qquad \text{A,B constants }, x\neq 0$$

or $$x^2y''+Axy'+by=0 \qquad \text{A,B constants}$$

is called Euler's equation.

2. Method of solution



Euler equation

$$y''+\frac{A}{x}y'+\frac{B}{x^2}y=0$$

let t = lnx (or $x=e^t$)

$$\bar{Y}(t)=y(x)=y(e^t)$$

$$y'(x)=\frac{dy}{dx}=\frac{d\bar{Y}}{dx}=\frac{d\bar{Y}}{dt}\times\frac{dt}{dx}=\frac{\bar{Y}'(t)}{x}$$

or

$$\bar{Y}'(t)=xy'(x) \text{----}(1)$$

$$y''(x)=\frac{d}{dx}(y'(x))$$

$$=\frac{d}{dx}\left(\frac{\bar{Y}'(t)}{x}\right)$$

$$=-\frac{\bar{Y}'(t)}{x^2}+\frac{1}{x}\times\frac{d\bar{Y}'(t)}{dx}$$

$$=-\frac{\bar{Y}'(t)}{x^2}+\frac{1}{x}\times\frac{d\bar{Y}'(t)}{dt}\times\frac{dt}{dx}$$

$$=\frac{1}{x^2}(\bar{Y}''(t)-\bar{Y}'(t))$$

Or

$$x^2 y'' = \overline{Y}''(t) - \overline{Y}'(t) \text{----(2)}$$

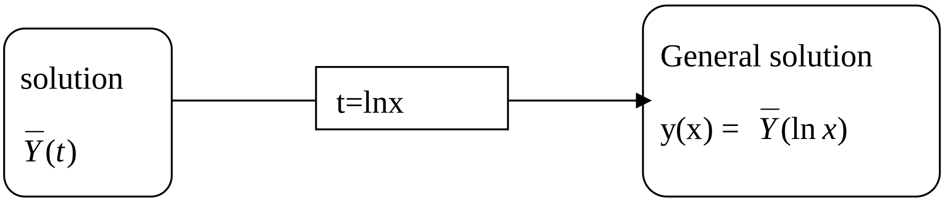
$$x^2 y'' + Axy' + by = 0 \text{----(3)}$$

Substituting Equations (1)&(2) into Euler Equation (3) leads to

$$\overline{Y}''(t) - \overline{Y}'(t) + A\overline{Y}'(t) + B\overline{Y}(t) = 0$$

$$\overline{Y}''(t) + (A-1)\overline{Y}'(t) + B\overline{Y}(t) = 0$$

which is constant coefficient homogeneous 2^{nd}-order linear Equation.



EX: $x^2 y'' + 2xy' - 6y = 0$ (A=2,B=-6)

(Sol):let t=lnx

$$\overline{Y}''(t) + (A-1)\overline{Y}'(t) + B\overline{Y}(t) = 0$$

$$\therefore \overline{Y}''(t) + \overline{Y}'(t) - 6\overline{Y}(t) = 0$$

characteristic equation is

$$\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda + 2) = 0$$

$$\lambda_1 = -3, \lambda_2 = 2$$

distinct real roots

general solution for $\overline{Y}(t)$

$$\overline{Y}(t) = c_1 e^{-3t} + c_2 e^{2t}$$

$$y(x) = c_1 e^{-3\ln x} + c_2 e^{2\ln x}$$
$$= c_1 x^{-3} + c_2 x^2$$

Remark:

D.E	functions as solutions
$y'' + Ay' + By = 0$	$e^{ax}, xe^{ax}, e^{ax}\cos bx, e^{ax}\sin bx$
$x^2 y'' + Axy' + By = 0$	$x^r, x^r \ln x, e^{px}\cos(q\ln x), e^{px}\sin(q\ln x)$

2.6 Nonhomogeneous linear Equation

2^{nd}-order nonhomogeneous linear D.E.

$y'' + p(x)y' + q(x)y = f(x)$

general solution

$y(x)=y_h(x)+y_p(x)$

$y_h(x)$=general solution of $y'' + p(x)y' + q(x)y = 0$

$y_p(x)$=particular solution of $y'' + p(x)y' + q(x)y = f(x)$

How to solve the particular solution

(1)Method of variation parameters

(2)Method of undermined coefficients

(3)Inverse operator

1. Method of variation of parameters

$y'' + p(x)y' + q(x)y = f(x)$

(1) Find $y_h(x)$

Solve $y'' + p(x)y' + q(x)y = 0$

Assume $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

(2) Find $y_p(x)$

$y'' + p(x)y' + q(x)y = f(x)$ ----(1)

Assume the particular solution

$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ ----(2) ,

v(x) and u(x)?

$\therefore y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$

Let $y_1u' + y_2v' = 0$ ----(3)

then $y_p' = uy_1' + vy_2'$ ----(4)

$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$ ----(5)

Substituting Eqs. (2), (4) and (5) in Eq. (1), we have

$u[y_1'' + py_1' + qy_1] + v[y_2'' + py_2' + qy_2] + y_1'u' + y_2'v = f(x)$

$y_1'u' + y_2'v' = f(x)$ ----(6)

From Eqs. (3)&(6), we have

$y_1u' + y_2v' = 0$

$y_1'u' + y_2'v' = f(x)$

two eqs for two unknows u', v'

By Cramer’s rule, the solution of u’ and v’ are

$$u' = \frac{-y_2 f(x)}{w(y_1, y_2)}$$

$$v' = \frac{y_1 f(x)}{w(y_1, y_2)}$$

By integration

$$u(x) = \int \frac{-y_2 f(x)}{w(y_1, y_2)} dx$$

$$v(x) = \int \frac{y_1 f(x)}{w(y_1, y_2)} dx$$

$$\therefore y_p(x) = uy_1 + vy_2$$

$$= -y_1 \int \frac{y_2 f(x)}{w} dx + y_2 \int \frac{y_1 f(x)}{w} dx$$

Ex: $y'' - 4y' + 4y = (x+1)e^{2x}$

General solution?

(Sol)

(1) Find $y_h(x)$

$$y'' - 4y' + 4y = 0$$

$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$ repeated roots

$$\lambda_1 = \lambda_2 = 2$$

general solution

$$y_h(x) = c_1 e^{2x} + c_2 x e^{2x}$$

(2)Find $y_p(x)$

Use the method of variation of parameters to find $y_p(x)$

$$y_p = u(x) y_1(x) + v(x) y_2(x)$$

$$w(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

$$u' = \frac{-y_2 f(x)}{w(y_1, y_2)} = \frac{-xe^{2x} \times (x+1)e^{2x}}{e^{4x}} = -x(x+1)$$

$$v' = \frac{y_1 f(x)}{w(y_1, y_2)} = \frac{e^{2x} \times (x+1)e^{2x}}{e^{4x}} = (x+1)$$

$$u = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$v = \frac{x^2}{2} + x$$

$$y_p = u(x) y_1(x) + v(x) y_2(x) = (x^3/6 + x^2/2)e^{2x}$$

(3) Find y(x)

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 x e^{2x} + (x^3/6 + x^2/2)e^{2x}$$

2. Method of undetermined coefficients

Non-homogeneous linear D.E. y''+ Ay'+ By = f(x)

<i> constant coefficients A, B

<ii> f(x) = (a) constant

(b) x^n

(c) $e^{\alpha x}$

(d) Cos βx, Sinβx

(e) Combinations of (a) to (d)

(1) Basic rule of method of solution $y_p(x)$

f(x)	$y_p(x)$
x^n	$x^n, x^{(n-1)}, \ldots, x, 1$
$e^{\alpha x}$	$e^{\alpha x}$
Cos βx	Cos βx, Sinβx
Sinβx	Cos βx, Sinβx
Sum of above functions	union of above functions
Product of above functions	union of Product of above functions

Ex: f(x) — $y_p(x)$

$8x^2-2x$ — ax^2+bx+c

$4e^{2x}$ — Ae^{2x}

-3Sin2x — aSin2x + bCos2x

$x^2 e^{5x}$ — $(ax^2+bx+c)\, e^{5x}$

e^{2x}Cos 2x — e^{2x} (aSin2x + bCos2x)

tan 3x — No $y_p(x)$ can be assumed!

(2) Correction rule

A. If a function in the solution $y_p(x)$ duplicates a function in the solution $y_h(x)$, the solution $y_p(x)$ must be modified.

B. If any $y_{pi}(x)$ contains terms that duplicate terms in $y_h(x)$, then $y_{pi}(x)$ must be multiplied by x^σ, where σ is the smallest positive integer that will eliminate the duplication.

Ex: $y'' + 2y' - 3y = 8e^x$

(i) $y_h(x) = C1\, e^{-3x} + C2\, e^{x}$ (y''+ 2y' - 3y = 0)

(ii) $y_p(x) = ?$

$f(x) = 8e^x$

$y_p(x) = Ae^x$ is a solution of $y_h(x)$, so it does not work.

$\therefore y_p(x) = A\,(x^\sigma)(e^x) = Axe^x$ (σ = 1)

Ex: $y'' - 6y' + 4y = 5e^{3x}$

(i) $y'' - 6y' + 4y = 0$

$y_h(x) = C1\, e^{3x} + C2\, xe^{3x}$

(ii) $y_p(x)$, $f(x) = 5e^{3x}$

Assume $y_p(x) = A\, e^{3x}$ (not work)

∴ we must assume

$y_p(x) = A(x^{\sigma})(e^{3x}) = Ax^2\, e^{3x} (\sigma = 2)$

(3) Possible forms of $y_p(x)$ for given $f(x)$

$f(x)$	$y_p(x)$
$p(x)$ $p(x)\, e^{\alpha x}$ $p(x)\{Sin\beta x, Cos\beta x\}$ $p(x)(e^{\alpha x})\{Sin\beta x, Cos\beta x\}$	$q(x)\, x^{\sigma}$ $q(x)\, x^{\sigma}\, e^{\alpha x}$ $x^{\sigma}[q(x)\, Cos\beta x + r(x)\, Sin\beta x]$ $x^{\sigma}\, e^{\alpha x}\, [q(x)\, Cos\beta x + r(x)\, Sin\beta x]$
$p(x) = a0 + a1x + \ldots\ldots + an\, x^n$ $q(x) = b0 + b1x + \ldots\ldots + bn\, x^n$ $r(x) = c0 + c1x + \ldots\ldots + cn\, x^n$	

Ex: $y'' + 9y = 4xSin(x)$

general solution $y(x) = ?$

Sol: (1) $y_h(x) = ?$

$y'' + 9y = 0$

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$y_h(x) = C1Cos3x + C2Sin3x$

(2) $y_p(x)$

$f(x) = 4xSin(x)$

$y_p(x) = x^{\sigma}[(ax+b)\, Cos3x + (cx+d)\, Sin3x]$

if $\sigma = 0$ bCos3x & dSin3x are solutions of $y_h(x)$

∴ if $\sigma = 1$ then eliminate the duplication

$y_p(x) = (ax^2+bx)\, Cos3x + (cx^2+dx)\, Sin3x$

Ex: $y'' + 4y = x + 2e^{-2x}$

Sol: (1) $y_h(x)$

$y_h(x) = C1\, Cos2x + C2\, Sin2x$

(2) $y_p(x)$

$$f(x) = x + 2e^{-2x} = f_1(x) + f_2(x)$$

$$y'' + 4y = f_1(x) = x$$

$$y_{p1}(x) = \frac{1}{4}x$$

$$y'' + 4y = f_2(x) = 2e^{-2x}$$

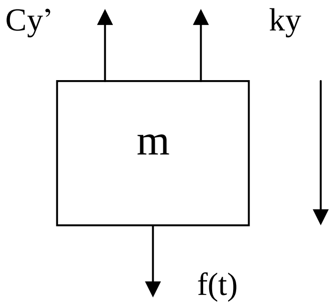
$$y_{p2}(x) = \frac{1}{4} e^{-2x}$$

$$y_p(x) = y_{p1}(x) + y_{p2}(x)$$

2.7 Application of 2nd-order D.E. to a mechanical system

1. Spring-damper-mass-system

Free-Body diagram of mass



Forces on mass

(A) external force f(t)

(B) spring force ky

(C) damping force cy'

Total external force acting on the mass is F(t) = f(t)-ky-cy

From Newton's 2nd law of motion, we have $\frac{d}{dt}(my) = F(t)$

Assume m=constant, so the system equation of motion becomes

$my'' = f(t) - ky - cy'$

$my'' + cy' + ky = f(t)$

or $y'' + \frac{c}{m}y' + \frac{k}{m}y = f(t)$ ----2nd-order D.E $(y'' + Ay' + By = f(t))$

(1) Unforced motion f(t)=0

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$$

characteristic equation

$$\lambda^2 + c/m\,\lambda + k/m\,y = 0$$

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$$

case1: c^2-4km>0, overdamping(相異實根)

$$\lambda_1 = -\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4km}$$

$$\lambda_2 = -\frac{c}{2m} - \frac{1}{2m}\sqrt{c^2 - 4km}$$

general solution

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$\lambda_1 \le 0 \ \& \ \lambda_2 \le 0$$

$$\lim y(t) = 0$$

$$t \to \infty$$

case2: c^2-4km=0, critical damping(重根)

$$\lambda_1 = \lambda_2 = \frac{-c}{2m}$$

$$y(t) = (c_1 + c_2 t)e^{-\frac{c}{2m}t}$$

case3: c^2-4km<0, underdamping(共軛虛根)

$$\lambda_1 = -\frac{c}{2m} + \frac{1}{2m}\sqrt{4km - c^2}\ \mathrm{i}$$

$$\lambda_2 = -\frac{c}{2m} - \frac{1}{2m}\sqrt{4lm - c^2} i$$

$$y(t) = e^{-\frac{c}{2m}t}(c_1 \cos(\frac{1}{2m}\sqrt{4km - c^2}t) + c_2 \sin\sqrt{4km - c^2}t)$$

(2) Forced motion f(t)$\neq$ 0

A .f(t)=Acos(wt)

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = \frac{A}{m}\cos wt$$

(i)y_h(t)

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0$$

unforced motion

Three cases :overdamping, critical damping and underdamping.

(ii)y_p(t)

$$y_p(t) = a\cos(wt) + b\sin(wt)$$

By method of undermined coefficients, we have

$$a = \frac{A(k - mw^2)}{(k - mw^2)^2 + w^2c^2}$$

$$b = \frac{Awc}{(k - mw^2)^2 + w^2c^2}$$

let

$$w_0 = \sqrt{\frac{k}{m}} \text{ (natural frequency)}$$

then

$$y_p(t) = \frac{mA({w_0}^2 - w^2)}{m^2({w_0}^2 - w^2)^2 + w^2c^2}\cos wt + \frac{Amc}{m^2({w_0}^2 - w^2)^2 + w^2c^2}\sin wt$$

B. Resonance

If damping c=0 then

$$y_p(t) = \frac{A}{m({w_0}^2 - w^2)}\cos wt$$

$$y_h(t) = c_1 \cos w_0 t + c_2 \sin w_0 t (\ddot{y} + \frac{k}{m} y = 0)$$

$$\therefore y(t) = y_h(t) + y_p(t)$$

$$= c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{A}{m({w_0}^2 - w^2)}\cos wt$$

$$asw \to w_0$$

$$y(t) \to \infty$$

2. Analogy with an Electrical Circuit (RLC circuit)

$$E(t) = L\frac{di}{dt} + Ri + \frac{q}{c}$$

$$i(t) = \frac{dq}{dc}$$

$$\therefore L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t)$$

or

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{c}i = E'(t)$$

In comparison with the equation of spring-damper-mass system

$$m\ddot{y} + c\dot{y} + ky = f(t)$$

Equivalence and Analogy between RLC circuit and mck system

Mechanical	Electrical quantity
M	L
C	R
K	1/C
Y	q(t)
$\dot{y}$	i(t)
f(t)	E(t)