Chapter 12

Vector Integral Calculus

12.1 Line integral

1. Curve in 3-space

(1) parametric equation

$C: x = x(t), y = y(t), z = z(t); a \le t \le b$

initial point $(x(a), y(a), z(a))$

final point $(x(b), y(b), z(b))$

Ex. Curve: $C_1 : x = 2\cos t, y = 2\sin t, z = 4, 0 \le t \le 2\pi$

Curve: $C_2 : x = 2\cos t, y = 2\sin t, z = 4, 0 \le t \le 4\pi$

（2）Continuous

X（t），Y（t），Z（t） are continuous on a$\le t \le b$

C_1： X＝2 cost ，Y＝2 sint ，Z＝4；0$\le t \le 2\pi$

∵ 2 cost ，2 sint ，4 continuous on 0$\le t \le 2\pi$

∴ C_1 is continuous

（3）differentiable

X（t），Y（t），Z（t）are differentiable

（4）Smooth （No sharp points or corners）

X'（t），Y'（t），Z'（t）are continuous on a$\le t \le b$ and not all zero

C_1： X＝2 cost ，Y＝2 sint ，Z＝4；0$\le t \le 2\pi$

X'＝-2 cost , Y'＝2 sint, Z'＝0 are continuous on 0$\le t \le 2\pi$

⇒Curve C_1 is smooth

（5）Position vector of curve

$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ ，a$\le t \le b$

For smooth curve， $\vec{R}(t) \ne 0$ and continuous

（6）Closed curve

Initial and terminal are the same

Ex： C_1 ：X＝4 cost ，Y＝4 sint ，Z＝4 ，0$\le t \le 3\pi$

C_2 ：X＝4 cost ，Y＝4 sint ，Z＝4 ，0$\le t \le 4\pi$

C_1 ：not closed curve

C_2 ：closed courve

2. Line Integrals of vector Field

（1） Defintion

The line integral of vector field $\vec{F}(x,y,z)$ over a smooth curve c is denoted $\int_c \vec{F}\bullet d\vec{R}$ and is defined by

$$\int_a^b \vec{F}(x(t),y(t),z(t))\cdot\vec{R}(t)\,\mathrm{dt}$$

where $\vec{R}(t)$ is position vector of a smooth curve c for $a\le t\le b$ ， and $\vec{F}$ is continuous at points on c.

$$\vec{F}\cdot d\vec{R}=\vec{F}\cdot\frac{d\vec{R}}{dt}dt$$

$$=\vec{F}\cdot\vec{R}'dt$$

Ex：Evaluate $\int_c \vec{F}\bullet d\vec{R}$

Where $\vec{F}=x\vec{i}-yz\vec{j}+e^z\vec{k}$

$c: x=t^3, y=-t, z=t; 0\le t\le 1$

Sol： $\vec{R}(t)=\ t^3\vec{i}-t\vec{j}+t\vec{k};0\le t\le 1$

$\vec{F}(t)=\ x\vec{i}-yz\vec{j}+e^z\vec{k}$

$=t^3\vec{i}-(-t)\cdot(t)\vec{j}+e^t\vec{k}$

$=t^3\vec{i}+t^2\vec{j}+e^t\vec{k}$

$$\therefore \int_c \vec{F}\bullet d\vec{R}=\int_0^1 \vec{F}\cdot\vec{R}'dt=\int_0^1 (t^3,t^2,e^t)\cdot(3t^2,-1,1)dt$$

$$=\int_0^1 (3t^5-t^2+e^t)dt$$

$$=-\frac{5}{6}+e$$

（2） Piecewise smooth curve （or Path）

A curve where position vector $\vec{R}(t)$ is continuous and tangent vector $\vec{R}'(t)$ is continuous and different from zero vector at all but possibly a finite number of values of t.

（3）line integral to piecewise smooth curve

$$\int_c \vec{F}\bullet d\vec{R}=\int_{c_1}\vec{F}\cdot d\vec{R}+\int_{c_2}\vec{F}\cdot d\vec{R}........+\int_{c_n}\vec{F}\cdot d\vec{R}$$

where c is a piecewise smooth curve, consisting of smooth curves

c_j ， j＝1,2,3,............. , n , and $\vec{F}$ is a continuous vector field on c.

（4）Physical Interpretation

Line integral of vector field $\int_c \vec{F} \bullet d\vec{R}$ can be interpreted as the work done by a force $\vec{F}$ in moving the particle over the path c.

$d\vec{R}$： displacement of particle

work＝$dU = \vec{F} \bullet d\vec{R}$

$U = \int dU = \int_c \vec{F} \bullet d\vec{R}$

Ex： Given ：Force $\vec{F} = \vec{i} - y\vec{j} + xyz\vec{k}$

curve c ：x=t, $y = t^2, z = t; 0 \le t \le 1$

Find： work done by force $\vec{F}$ in moving a particle along curve

Sol： work＝$\int_c \vec{F} \bullet d\vec{R} = \int_0^1 \vec{F} \cdot \vec{R}'(t)dt$

$\vec{R}(t) = t\vec{i} - t^2\vec{j} + t\vec{k}$

$\vec{R}'(t) = \vec{i} - 2t\vec{j} + \vec{k}$

$\vec{F}(x(t), y(t), z(t)) = \vec{i} - y\vec{j} + xyz\vec{k}$

$= \vec{i} + t^2\vec{j} - t^4\vec{k}$ on c

$$\therefore \int_0^1 \vec{F} \cdot \vec{R}'(t)dt = \int_0^1 (1 - 2t - t^4)dt = \frac{3}{10}$$

（5） Properties

i. $\int_c (\vec{F} + \vec{G}) \cdot d\vec{R} = \int_c \vec{F} \cdot d\vec{R} + \int_c \vec{G} \cdot d\vec{R}$

ii. $\int_c (\vec{F}) \cdot d\vec{R} = \int_c \vec{F} \cdot d\vec{R}$ α :any number

iii. $\int_c \vec{F} \bullet d\vec{R} = -\int_{-c} \vec{F} \cdot d\vec{R}$

3. Line Integrals of scalar field

$$\int_c f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$$
$$= \int_a^b \left[f(x(t),y(t),z(t))\frac{dx}{dt} + g(x(t),y(t),z(t))\frac{dy}{dt} + h(x(t),y(t),z(t))\frac{dz}{dt} \right] dt$$

where f, g and h are continuous functions on curve c having parametric functions x=x(t),y=y(t),z=z(t) for $a \le t \le b$.

If

$\vec{F}(x,y,z) = f\vec{i} + g\vec{j} + h\vec{k}$ $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$

then

$$\vec{F}\cdot d\vec{R} = fdx + gdy + hdz$$

so

$$\int_c (fdx + gdy + hdz = \int_c \vec{F}d\vec{R})$$

Another way of line Integral of a vector field

4. Line integral writ Arc length

(1)Definition

$$\int_c f(x.y.z)ds = \int_a^b f(x(s), y(s), z(s))ds$$
$$= \int_a^b f(x(t), y(t), z(t))\left\|\vec{R}(t)\right\|dt$$

Where c is smooth curve and f(x,y,z) is real-valued function
If

$$c = \vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad for \quad a \le t \le b$$

then arc length s(t) along c is

$$s(t) = \int_a^t \left\|\vec{R}'\right\|(\zeta)d\zeta$$
$$\therefore \frac{ds}{dt} = \left\|\vec{R}'\right\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$
$$Hence \int_c f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\frac{ds}{dt}dt$$
$$= \int_a^b f(x(t), y(t), z(t)\left\|\vec{R}'\right\|)dt$$

(2) Engineering Application

(i) mass of a wire

$$mass = \int_c \delta(x,y,z)ds = \int_a^b \delta(x(t),y(t),z(t)\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2})dt$$

where $\delta(x,y,z)$ *is the density of the materiral*

(ii) center of mass of the wire

$$\bar{x} = \frac{1}{m}\int_c x\delta(x,y,z)ds$$

$$\bar{y} = \frac{1}{m}\int_c y\delta(x,y,z)ds$$

$$\bar{z} = \frac{1}{m}\int_c z\delta(x,y,z)ds$$

$$m = \int_c \delta(x,y,z)ds$$

12.2 Green's Theorem

1. line integral along simple closed cures

$$\oint_c \vec{F}\cdot d\vec{R} = \oint_c f(x,y)dx = g(x,y)dy$$

(1) positive orientation of closed curve C

c:x=x(t)，y= y(t)， $a \le t \le b$ in the plane

(x(t)，y(t)) moves around C counterclockwise.

(2) simple curve

same point cannot be on the curve for different values of parameter.

(3) simple closed curve

A closed curve where initial and terminal point are the only point that coincide for different values of the parameter.

2. Green Theorem on the plane

(1) Definition

$$\int_c \vec{F} \cdot \vec{R} = \int_c f dx + g dy$$

$$= \iint_d [\frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y}] dA$$

Where C = a simple closed positively oriented path

D = c U interior of c

$$\vec{F}(x, y) = f(x, y)\vec{i} + g(x, y)\vec{j}$$

$$and \ f, g, \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} \ are \ continous \ on \ D$$

(2)Applications

(i)Convert a difficult integration into an easier one

EX: Force $\vec{F} = (y - x^2 e^x)\vec{i} + (\cos y^2 - x)\vec{j}$

$$f(x, y) = y - x^2 e^x$$

$$g(x, y) = \cos y^2 - x$$

$$\frac{\partial f}{\partial y} = 1, \ \frac{\partial g}{\partial x} = -1$$

From Greens Theorem

$$\oint = \vec{F} \cdot d\vec{R} = \iint_d [\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}] dA = -2 \iint_d dA = -4$$

(ii)Evaluate general line integral

$$I : \oint_c 2x \cos(2y) dx - 2x^2 \sin(2y) dy$$

$$C : any \ poisitively \ oriented \ path$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

$$\therefore I = \iint_d [\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}] dA = 0$$

12.3 Independence of path and potential Theory

1. Conservative Vector Field

(1)Definition

$$Vector \ field \ \vec{F} \ is \ called \ conserralive \ if \ \vec{F} = \nabla \phi$$

$$Scalar \ field \ \phi \ is \ called \ a \ potential \ (function) \ for \ \vec{F}$$

(2)Line intergral

$$\vec{F} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j}$$

$$\vec{R}' = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

$$\therefore \vec{F}(x(t), y(t)) \cdot \vec{R}'(t) = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} = \frac{d\phi(x(t), y(t))}{dt}$$

$$\Rightarrow \int_C \vec{F} \, d\vec{R} = \int_a^b \vec{F}\,(X(t)Y(t))\,\vec{R}\,(t)dt = \int_a^b \frac{d\phi}{dt}dt = \int_a^b d\phi = \phi\,(X(t),Y(t))\Big|_b^a = \phi\,(X(b),Y(b)) - \phi\,(X(a),Y(a))$$

2 Independence of path

(1) Definition

$\int_C \vec{F} \, d\vec{R}$ B independent of path in D if the integral has the same value over any path in D for Pt,Po to Pt,P1

(2) line integral closed path if $\vec{F} = \nabla\phi$,

then $\int_C \vec{F} \, d\vec{R}$ is independent of path in D . If C is a closed path in D,then

$$\int_C \vec{F} \, d\vec{R} = 0$$

(3) Criferion of a conservatre field vecfor field

$\vec{F} = f(x,y)\,\vec{i} + g(x,y)\,\vec{j}$ is conservativers on a region R if and only if , at (x,y) in R

$$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial y}$$

$\vec{F}$ d $\vec{R}$ = f(x,y)dx+ g(x,y)dy

= d ϕ

$$=\frac{\partial y}{\partial x}dx=\frac{\partial \varphi}{\partial y}dy$$

$$\therefore \frac{\partial \varphi}{\partial x}=f,\ \frac{\partial \varphi}{\partial y}=g$$

$$\frac{\partial^2 \varphi}{\partial y \partial x}=\frac{\partial \varphi}{\partial y \partial x}=\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$$

12-4 Surface Integrals

I surfaces (having area but no volume)

(1) parametine equation

(i)X=X(u,v) , Y=Y(u,v) ,Z=Z(u,v)

U , V two independent variables

Ex.

(1) cylinder $X^2+Y^2=a^2,-1\le z\le 1$,

r(u , v) = a cosu $\vec{i}$ +a sinu $\vec{j}$

+v $\vec{k}$

$$0 \le u \le 2\pi, -1 \le v \le 1$$

(2) sphere

$$x^2 + y^2 + z^2 = a^2$$

$r(u,v) = a\cos v\cos u\ \vec{i} + a\cos i\sin u\ \vec{j} + a\sin v\ \vec{k}$

$$0 \le u \le 2\pi, \frac{N}{2} \le v \le \frac{\pi}{2}$$

(3) cone

$$z = \sqrt{x^2 + y^2} \quad 0 \le z \le h$$

$r(u,v) = u\cos v\ \vec{i} + u\cos i\ \vec{j} + u\ \vec{k}$

$$0 \le u \le H, 0 \le v \le 2\pi$$

(ii) Z=S (X ,Y)

Lows of pts （x , y ,s(x ,y)）

Ex. Surface

$$z = \sqrt{4 - x^2 - y^2} \quad x^2 + y^2 \le 4, \; z \le 0$$

Sol：square the equation

=> $x^2 + y^2 + z^2 = 4$

=>hemisphere of radins 2 with origin(0,0,0)

Ex. Surface

$$z = \sqrt{x^2 - y^2} \quad x^2 + y^2 \le 8$$

(2)position rector of a surface Z=S (X , Y)

$$\vec{R}(X1 , Y1)= X\vec{i} +Y\vec{j} +S(x ,y)\vec{k}$$

(3)simple surface

The position rector $\vec{R}$ (x , y) does'nt return to the same point for different (x , y) .

$$\vec{R}(X1 , Y1)=\vec{R}(X2,Y2)\Rightarrow X1=X2\ Y1=Y2$$

(4)Normal rector $\vec{N}$ to a surface

A rector orthogonal to each rector on the tangent plane at Po

Write q(x , y, z) = s(x , y) z

So z = s(x , y) is level surface φ (x , y , z) = 0

Gradient of φ

$$\partial = \frac{\partial \varphi}{\partial x}\vec{i} = \frac{\partial \varphi}{\partial y}\vec{j} = \frac{\partial \varphi}{\partial z}\vec{k}$$

$$= \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} - \vec{k}$$

$$= \vec{N} = \text{normal}$$

(5)smoth surface

A surface Σ is simple and has a confinuons , nonzero normal rector at every pt.

(6)piecenis smooth

A surface consists of a finite number of smooth surface.

Ex. Sphere surface ：smooth

Surface a oube ： pieconise smooth

(7)surface area

The area of a smooth surface Σ given by z：s(x, y) is given by A(Σ)＝

$$\int\int_0 \sqrt{1 + sx^2 + sy^2}\,dxdy$$

$$= \int\int_0 \left\| \overline{\quad}_x \rightarrow (x \ , y) \right\| da$$

＝Double integral of the lergth of the normal rector.

2. surface Integral of a function of three variables

Let f(x,y,z) be a function of three variables defined over a region of space containing surface Σ then the surface integral of f(x,y,z) over a smooth surface Σ B olefound by $\iint f(x,y,z)d\delta$

$$= \iint f(x.y.s(x,y))\sqrt{1+sx^2} + sy^2 dA$$

$$= \iint f(x.y.z(x,y))\sqrt{1+zx^2} + zy^2 dA$$

3. some use of surface integrals

1. surface area

If f(x,y,z)=1

Then $\iint d\delta = \iint \sqrt{1+sx^2+sy^2}\,dA = A(\sum \ )$

2.Mass and couteo of mass of a shell

$$m = \iint \delta(x,y,z)d\sigma = \iint \delta(x.y.\delta(x,y))\sqrt{1+sx^2} + sy^2 dA$$

$$X = \frac{1}{m}\iint x\delta(x,y,z)d\sigma$$

$$Y = \frac{1}{m}\iint y\delta(x,y,z)d\sigma$$

Z= $\frac{1}{m}\iint z\delta(x,y,z)d\sigma$

12.6 vector forms of green's theorem

1. Green;s theorem in 2-space

$$\oint (fdx+gdy)=\iint\left[\frac{\partial g}{\partial z}\frac{\partial f}{\partial y}\right]dA$$

$$\forall F=\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

(1) let F= gi-fj

so $\oint fdx+gdy=\oint\left[f\frac{dx}{ds}+g\frac{dy}{ds}\right]ds=\oint FNds$

(2) let F(x,y,z)=f(x,y)i+g(x,y)j+0k

$$\forall\times F=\left[\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right]\text{K}$$

curl F=

$(J\times F)\circ K=\left[\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right]$ and $\vec{F}\vec{I}ds=\left[f\dot{i}+g\vec{j}\right]\cdot\left[\frac{dx}{ds}\dot{i}+\frac{dy}{ds}\vec{j}\right]ds=fdx+gdy$

so $\oint_c fdx+gdy=\oint_c\overline{F\vec{J}}ds$ Hence $\oint_c\overline{F\vec{J}}ds=\iint_D\left(\forall+\vec{F}\right)\cdot\vec{K}dA$

2. Green Theorem in 3-space

(1) Gauce;s integral theorem

close cure c => close surface Σ lore

integral -> surface ontegroul

$\iint_\delta\vec{F}\vec{N}dg=\iiint_n\forall\vec{F}=\iiint_N dn\vec{F}dv$ surface integral <->triple integral

(3) stoke integral theorem

closed curve C-> a curve in 3 space flat surface D in xy plane ->a surface

Σ with unit normal $\vec{I}$

$$\oint_c\vec{F}\vec{T}ds=\iint_\Sigma\left(\forall\times\vec{F}\right)\cdot\vec{N}ds$$

=> $\int_C \overrightarrow{F}\, d\,\overset{\Delta}{R} = \int_a^b \overrightarrow{F}\, X(t)Y(t)\, \overrightarrow{R}\,(t)dt$

$$= \int_a^b \frac{d\phi}{dt} dt = \int_a^b d\phi = \phi \text{ (X(t),Y(t)) } \int_b^a = \phi \text{ (X(b),Y(b)) } \phi$$

(X(a),Y(a))

2 Independence of path

(1) Definition

$\int_C \vec{F} \, d \vec{R}$ B independent of path in D if the integral has the same value over any path in D for Pt,Po to Pt,P1

(2) line integral closed path if $\vec{F} = \nabla\phi$,

then $\int_C \vec{F} \, d \vec{R}$ is independent of path in D . If C is a closed path in D,then

$$\int_C \vec{F} \, d \vec{R} = 0$$

(3) Criferion of a conservatre field vecfor field

$\vec{F} = f(x,y) \vec{i} + g(x,y) \vec{j}$ is

conservativers on a region R if and only if , at (x,y) in R

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

$$\vec{F} \, d \vec{R} = f(x,y)dx + g(x,y)dy$$

$$= d\phi$$

$$= \frac{\partial y}{\partial x} dx = \frac{\partial \varphi}{\partial y} dy$$

$$\therefore \frac{\partial \varphi}{\partial x} = f, \ \frac{\partial \varphi}{\partial y} = g$$

$$\frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial \varphi}{\partial y \partial x} = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

12-4 Surface Integrals

I surfaces (having area but no volume)

(1) parametine equation

(i)X=X(u,v) , Y=Y(u,v) ,Z=Z(u,v)

U , V two independent variables

Ex.

(1) cylinder $X^2 + Y^2 = a^2, -1 \le z \le 1$,

r(u , v)＝a cosu $\overrightarrow{i}$ +a sinu $\overrightarrow{j}$

+v $\overrightarrow{k}$

$$0 \le u \le 2\pi, -1 \le v \le 1$$

(2) sphere

$$x^2 + y^2 + z^2 = a^2$$

r(u ,v) = a cos v cos u $\overrightarrow{i}$ +a cos i sin

u $\overrightarrow{j}$ +a sin v $\overrightarrow{k}$

$$0 \le u \le 2\pi , \frac{N}{2} \le v \le \frac{\pi}{2}$$

(3) cone

$$z = \sqrt{x^2 + y^2} \quad 0 \le z \le h$$

r(u ,v) = u cos v $\overrightarrow{i}$ +u cos

i $\overrightarrow{j}$ +u $\overrightarrow{k}$

$$0 \le u \le H, 0 \le v \le 2\pi$$

(ii) Z=S (X ,Y)

Lows of pts （x , y ,s(x ,y)）

Ex. Surface

$$z = \sqrt{4 - x^2 - y^2} \quad x^2 + y^2 \le 4 , z \le 0$$

Sol：square the equation

=> $x^2 + y^2 + z^2 = 4$

=>hemisphere of radins 2 with origin(0,0,0)

Ex. Surface

$$z = \sqrt{x^2 - y^2} \quad x^2 + y^2 \le 8$$

(2)position rector of a surface Z=S (X , Y)

$\overrightarrow{R}$ (X1 , Y1)= X $\overrightarrow{i}$ +Y $\overrightarrow{j}$ +S(x ,y) $\overrightarrow{k}$

(3)simple surface

The position rector $\vec{R}$ (x , y) does'nt return to the same point for different (x , y) .

$\vec{R}$(X1 , Y1)= $\vec{R}$(X2,Y2)=>X1=X2 Y1=Y2

(4)Normal rector $\vec{N}$ to a surface

A rector orthogonal to each rector on the tangent plane at Po

Write q(x , y, z)＝s(x , y) z

So z＝s(x , y) is level surface φ (x , y , z)＝0

Gradient of φ

$$\partial = \frac{\partial \varphi}{\partial x}\vec{i} = \frac{\partial \varphi}{\partial y}\vec{j} = \frac{\partial \varphi}{\partial z}\vec{k}$$

$$= \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} - \vec{k}$$

$$= \vec{N} = \text{normal}$$

(5)smoth surface

A surface Σ is simple and has a confinuons , nonzero normal rector at every pt.

(6)piecenis smooth

A surface consists of a finite number of smooth surface.

Ex. Sphere surface ：smooth

Surface a oube ： pieconise smooth

(7)surface area

The area of a smooth surface Σ given by z：s(x, y) is given by A(Σ)＝

$$\int\int_0 \sqrt{1 + sx^2 + sy^2}\,dxdy$$

$$= \int \int_0 \left\| \overline{\quad}_x \to (x \quad , y) \right\| da$$

$=$ Double integral of the lergth of the normal rector.