

CHAPTER 11

Vector Differential Calculus

11.1 Vector function of one variable

1. Definition

A vector function of one variable , each component of which is a vector function of the same variable

$$\vec{F}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$\vec{F}(t) = \cos(t)\vec{i} + 2t^2\vec{j} + 3t\vec{k}$$

$$\vec{F}(0) = \vec{i}$$

$$\text{EX: } \vec{F}(2\pi) = \vec{i} + f\pi^2\vec{j} + b\pi\vec{k}$$

⋮

are vector

2. Continous

A vector function is continous of its component function are continuous at all t for where it is defined.

$$\text{EX: } \vec{F}(t) = \cos t\vec{i} + 2t^2\vec{j} + 3t\vec{k}$$

$$\begin{cases} x(t) = \cos t \\ y(t) = 2t^2 \\ z(t) = 3t \end{cases} \quad \text{continuous for all } t$$

$\therefore \vec{F}(t)$ is continuous for all t.

$$\text{EX: } \vec{G}(t) = \frac{1}{x-1}\vec{i} + \ln t\vec{k}$$

$$x(t) = \frac{1}{x-1} \quad \text{not continuous at } t=1$$

$$z(t) = \ln t \quad \text{continuous } t>0$$

$\therefore \vec{G}(t)$ is continuous for $t>0$ with $t \neq 1$

3.Derivative

$$\begin{aligned} \vec{F}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\vec{F}(t + \Delta t) - \vec{F}(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \vec{i} + \lim_{\Delta t \rightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} \vec{j} + \lim_{\Delta t \rightarrow 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} \vec{k} \\ &= x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} \end{aligned}$$

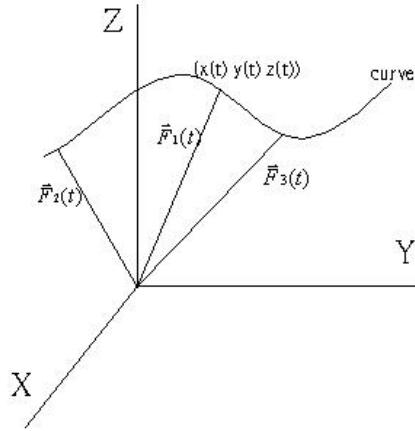
A vector function is differential of it has a derivative for are t for which it is defined.

$$\text{EX: } \vec{G}(t) = \frac{1}{t-1}\vec{i} + \ln t\vec{k} \quad \text{differentiable?}$$

Sol: $\vec{G}(t)$ is differentiable for positive t different from 1.

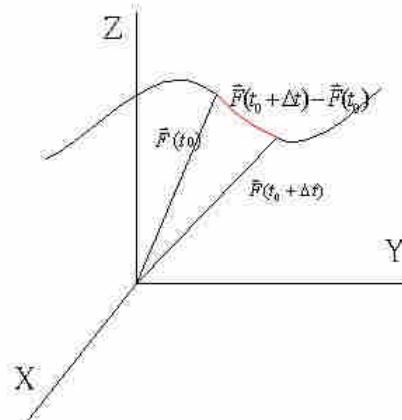
$$\vec{G}(t) = \frac{1}{(t-1)^2} \vec{i} + \frac{1}{t} \vec{k}$$

4. position vector of a curve



$\vec{F}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is called a position vector of the curve.

5. Tangent vector for a curve



$$\begin{aligned}\therefore \frac{\vec{F}(t_0 + \Delta t) - \vec{F}(t_0)}{\Delta t} &= \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} \vec{i} + \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t} \vec{j} + \frac{z(t_0 + \Delta t) - z(t_0)}{\Delta t} \vec{k} \\ \lim_{\Delta t \rightarrow 0} \frac{\vec{F}(t_0 + \Delta t) - \vec{F}(t_0)}{\Delta t} &= x'(t_0) \vec{i} + y'(t_0) \vec{j} + z'(t_0) \vec{k} \\ &= \vec{F}'(t)\end{aligned}$$

= tangent vector to the curve at pt. $(x(t_0) y(t_0) z(t_0))$

$$\therefore \vec{F}'(t) = x'(t_0) \vec{i} + y'(t_0) \vec{j} + z'(t_0) \vec{k}$$

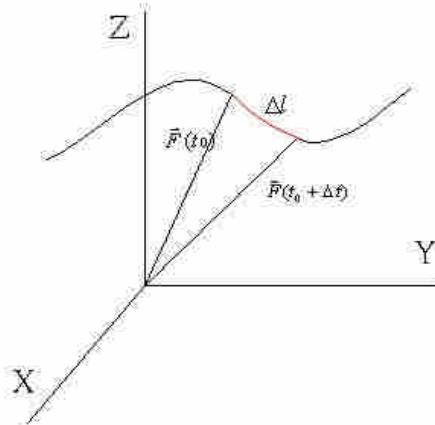
6. Length of curve

$$\vec{F}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad a \leq t \leq b$$

$$\text{Length} = \int_a^b \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2} dt$$

$$= \int_a^b \|\vec{F}(t)\| dt$$

Integral of the length of tangent vector



$$\therefore \Delta l = \|\vec{F}(t_0 + \Delta t) - \vec{F}(t_0)\| = \left\| \frac{\vec{F}(t_0 + \Delta t) - \vec{F}(t_0)}{\Delta t} \right\| \cdot \Delta t$$

If $\Delta t \rightarrow 0$

$$\text{Then } dl = \lim_{\Delta t \rightarrow 0} \left\| \frac{\vec{F}(t_0 + \Delta t) - \vec{F}(t_0)}{\Delta t} \right\| dt = \|\vec{F}(t)\| dt$$

$$\text{Length} = \int_a^b \|\vec{F}(t)\| dt$$

EX: circular helix (螺旋線)

$$x = \cos t \quad y = \sin t \quad z = 5/3 \quad -4\pi \leq t \leq 4\pi$$

$$\text{sol: position vector: } \vec{F}(t) = \cos t \vec{i} + \sin t \vec{j} + \frac{5}{3} \vec{k}$$

$$\text{tangent vector: } \vec{F}'(t) = -\sin t \vec{i} + \cos t \vec{j} + \frac{1}{3} \vec{k}$$

$$\therefore \|\vec{F}(t)\| = \frac{\sqrt{10}}{3}$$

$$\text{Length} = \int_{-4\pi}^{4\pi} \frac{\sqrt{10}}{3} dt = \frac{8}{3}\pi\sqrt{10}$$

7. Length function of a curve ($a \leq t \leq b$)

$$S(t) = \int_a^t \|\bar{F}'(\xi)\| d\xi \quad (\text{b} \rightarrow \text{t})$$

If $\bar{F}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ for $a \leq t \leq b$

And $x'(t) = y'(t) = z'(t)$ are continuous

$\therefore 0 \leq S(t) \leq 1$ as $a \leq t \leq b$

(1) position vector $\bar{G}(s)$

$$\bar{G}(s) = \bar{F}(t(1)) = x(t(1))\bar{i} + y(t(1))\bar{j} + z(t(1))\bar{k}$$

(2) Tangent vector $\bar{G}'(s)$

$$\bar{G}'(s) = \frac{d\bar{F}(t(1))}{ds} = \frac{d\bar{F}}{dt} \frac{dt}{ds} = \left(\frac{\bar{F}'}{\frac{ds}{dt}} \right)$$

$$\therefore \|\bar{G}'(s)\| = \frac{\|\bar{F}'(t)\|}{\left| \frac{ds}{dt} \right|}$$

where

$$\frac{ds}{dt} = \frac{d}{dt} \int_a^t \|\bar{F}'(\xi)\| d\xi = \|\bar{F}'(t)\|$$

so

$$\|\bar{G}(s)\| = \frac{\|\bar{F}'(t)\|}{\|\bar{F}'(t)\|} = 1$$

$\bar{G}(s)$ is a unit tangent vector.

8. Differentiation Rules

$$(1) \text{ Linearity } (\bar{F}(t) + \bar{G}(t))' = \bar{F}'(t) + \bar{G}'(t)$$

$$(2) \text{ Scalar product } (f(t)\bar{F}(t))' = f'(t)\bar{F}(t) + f(t)\bar{F}'(t)$$

$$(3) \text{ Dot product } (\bar{F}(t) \bullet \bar{G}(t))' = \bar{F}'(t) \bullet \bar{G}'(t)$$

$$(4) \text{ Cross product } (\bar{F}(t) \times \bar{G}(t))' = \bar{F}'(t) \times \bar{G}'(t)$$

$$(5) \text{ Chain rule } (\bar{F}(f(t)))' = \frac{d\bar{F}}{df} \frac{df}{dt} = f'(t)\bar{F}'(f(t))$$

11.2 Velocity, Acceleration, Curvature and Torsion

1. Motion of a particle on a curve

(1) Position Vector of particle P

$$\bar{F}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

(2) Velocity

$$\bar{V}(t) = \frac{d\bar{F}(t)}{dt} = x'(t)\bar{i} + y'(t)\bar{j} + z'(t)\bar{k}$$

If $\bar{V}(t) \neq 0$, $\bar{V}(t)$ is a tangent vector to the curve

(3) Acceleration

$$\bar{a} = \frac{d\bar{V}}{dt} = \bar{V}' = \bar{F}'' = x''(t)\bar{i} + y''(t)\bar{j} + z''(t)\bar{k}$$

(4) Distance traveled along the curve

$$Distance = s(t) = \int_a^t \|\bar{F}'(t)\| dt = \int_a^t \|\bar{V}(t)\| dt$$

(5) Speed (Speed = magnitude of velocity)

$$\therefore v(t) = \|\bar{V}(t)\|$$

$$\therefore s(t) = \int_a^t \|\bar{V}(t)\| dt$$

$$\therefore \frac{ds(t)}{dt} = \frac{d}{dt} \int_a^t \|\bar{V}(t)\| dt = \|\bar{V}(t)\| = v$$

rate of change of distance traveled along the curve

2. Curvature and Torsion of a curve

(1) Unit tangent vector $\bar{T}(t)$

$$\bar{T}(t) = \frac{\bar{F}'(t)}{\|\bar{F}'(t)\|} = \frac{\bar{F}'(t)}{\left(\frac{ds}{dt}\right)}$$

$$s(t) = \int_a^t \|\bar{F}'(\zeta)\| d\zeta : \text{Length function of a curve}$$

(2) Curvature K

The curvature K of a curve is the magnitude of the rate of change of the tangent vector with respect to one length along the curve

$$x(s) = \left\| \frac{d\bar{T}}{ds} \right\|$$

A.K quantifies the amount of bending of a curve at a point

B.The greater K is, the sharper the curve bends at the point

$$K(t) = \left\| \frac{d\bar{T}}{ds} \right\| = \left\| \frac{d\bar{T}}{dt} \frac{dt}{ds} \right\| = \left\| \frac{d\bar{T}}{dt} \right\| \left\| \frac{ds}{dt} \right\| = \frac{\left\| \frac{d\bar{T}}{dt} \right\|}{\left\| \frac{ds}{dt} \right\|} = \frac{\|\bar{T}'(t)\|}{\|\bar{F}'(t)\|}$$

Ex: Given: Straight line $\bar{F}'(t) = (a + bt)\bar{i} + (c + dt)\bar{j} + (e + ht)\bar{k}$ find K

Sol: $\bar{F}'(t) = b(t)\bar{i} + d(t)\bar{j} + h(t)\bar{k}$

$$\text{Unit tangent } \vec{T} = \frac{\vec{F}'(t)}{\|\vec{F}'(t)\|} = \frac{1}{\sqrt{b^2 + d^2 + h^2}} (b\vec{i} + d\vec{j} + h\vec{k})$$

$$\therefore \vec{T}'(t=0)$$

$$\therefore K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{F}'(t)\|} = 0$$

Ex: Given: A circle of radius 4 with origin in plane $y = 3$

Find: K of the give circle

Sol: position vector of a circle

$$\vec{F}(t) = 4 \cos t \vec{i} + 3 \vec{j} + 4 \sin t \vec{k} \quad 0 \leq t \leq 2\pi$$

$$\vec{F}'(t) = -4 \sin t \vec{i} + 4 \cos t \vec{k}$$

$$\vec{T}(t) = \frac{\vec{F}'(t)}{\|\vec{F}'(t)\|} = \frac{-4 \sin t \vec{i} + 4 \cos t \vec{k}}{4} = -\sin t \vec{i} + \cos t \vec{k}$$

$$\Rightarrow \vec{T}'(t) = -\cos t \vec{i} - \sin t \vec{k}$$

$$\therefore K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{F}'(t)\|} = \frac{\sqrt{(-\cos t)^2 + (-\sin t)^2}}{\sqrt{(-4 \sin t)^2 + (4 \cos t)^2}} = \frac{1}{4} = \frac{1}{r}$$

(3) Unit normal vector $\vec{N}(s)$

$$\vec{N}(s) = \frac{\vec{T}'(s)}{K(s)} \quad K(s) \neq 0$$

(i) $\|\vec{N}(s)\| = 1$ $\vec{N}(s)$: unit vecotr

(ii) $\vec{N}(s)$ orthogonal to unit tangent vector $\vec{T}(s)$

[Proof] $\because \|\vec{T}(s)\| = 1$

$$\|\vec{T}(s)\|^2 = \vec{T}(s) \bullet \vec{T}(s) = 1$$

$$\therefore (\vec{T}(s) \bullet \vec{T}(s))' = \vec{T}'(s) \bullet \vec{T}(s) + \vec{T}(s) \bullet \vec{T}'(s)' = 2\vec{T}'(s) \bullet \vec{T}(s) = 0$$

$\therefore \vec{T}'(s)$ orthogonal to $\vec{T}(s)$

$$\therefore \vec{N}(s) = \frac{\vec{T}'(s)}{K(s)} \therefore \vec{N}(s)$$
 orthogonal to $\vec{T}(s)$

(4) Tangential and Normal components of acceleration

$$\vec{a} = a_T \vec{T} + a_N \vec{N} = \frac{d\vec{v}}{dt} \vec{T} + K\vec{v}^2 \vec{N}$$

[proof] Look at $\vec{T}(t) = \frac{\vec{F}(t)}{\|\vec{F}(t)\|} = \frac{\vec{v}(t)}{v(t)}$

$$\therefore \vec{V}(t) = V(t)\vec{T}(t)$$

$$\vec{a} = \frac{d\vec{V}}{dt} = [V(t)\vec{T}(t)]' = V'(t)\vec{T}(t) + V(t)\vec{T}'(t)$$

$$= \frac{dV}{dt}\vec{T}(t) + v(t)\frac{d\vec{T}}{ds}\frac{ds}{dt}$$

$$= \frac{dV}{dt}\vec{T}(t) + V^2(t)\frac{d\vec{T}}{ds}$$

$$\vec{N}(s) = \frac{\vec{T}'(s)}{K(s)} \quad \therefore \frac{d\vec{T}}{ds} = K(s)\vec{N}(s)$$

$$\therefore \vec{a} = \frac{dV}{dt}\vec{T} + KV^2\vec{N} \quad \left(\begin{array}{l} a_T = \frac{dV}{dt} \\ a_N = KV^2 = \frac{V^2}{\rho}, K = \frac{1}{\rho} \end{array} \right)$$

$$= a_T\vec{T} + a_N\vec{N}$$

(5) Alternative formula of curvature

$$K = \frac{\|\vec{F}' \times \vec{F}''\|}{\|\vec{F}'\|^3}$$

(6) Binormal vector \vec{B}

A unit vector alternative to the plane determined by unit tangent \vec{T} and unit normal \vec{N}

$$\vec{B} = \vec{T} \times \vec{N} \quad (\vec{B}, \vec{T}, \vec{N} \text{ 均兩兩互相垂直})$$

(7) Frenet Formula

$$\begin{cases} \frac{d\vec{T}}{ds} = K\vec{N} \\ \frac{d\vec{N}}{ds} = -K\vec{T} + \tau\vec{B} \\ \frac{d\vec{B}}{ds} = -\tau\vec{N} \end{cases}$$

Here τ (s) is the “torsion” of curve at the point $(x(s), y(s), z(s))$, which measure the amount a curve twists

11.3 Vector Field and Streamlines

1. Vector Field 向量場

(1) Vector Field in the plane $\vec{G}(x, y)$. A vector has two component functions of variables

$$\vec{G}(x, y) = g_1(x, y)\vec{i} + g_2(x, y)\vec{j}$$

two variables x,y

(a) $\vec{G}(x, y)$ is a vector in x-y plane

(b) $\vec{G}(x, y)$ is represent as an arrow from (x,y)

$$\text{Ex: } \vec{G}(x, y) = xy\vec{i} + (x - y)\vec{j}$$

$$\therefore \vec{G}(0,0) = 0, \vec{G}(1,0) = \vec{j}, \vec{G}(0,2) = -2\vec{j}$$

$$\vec{G}(2,2) = -4\vec{i}, \vec{G}(1,-3) = -3\vec{i} + 4\vec{j}$$

(2)Vector field in 3-space

A vector has three components of three variables

$$\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$$

$\vec{F}(x, y, z)$ is represented by drawing the vector

$\vec{F}(x, y, z)$ as an arrow in 3-space from point (x,y,z)

(3)Partial derivative

$$\vec{G} = g_1\vec{i} + g_2\vec{j}$$

$$\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\therefore \frac{\partial \vec{G}}{\partial x} = \frac{\partial g_1}{\partial x}\vec{i} + \frac{\partial g_2}{\partial x}\vec{j}$$

$$\frac{\partial \vec{G}}{\partial y} = \frac{\partial g_1}{\partial y}\vec{i} + \frac{\partial g_2}{\partial y}\vec{j}$$

$$\frac{\partial \vec{F}}{\partial z} = \frac{\partial f_1}{\partial z}\vec{i} + \frac{\partial f_2}{\partial z}\vec{j} + \frac{\partial f_3}{\partial z}\vec{k}$$

2.Streamlines 流線

(1)Definition

The curve in Fare streamlines of the vector field $\vec{F}(x, y, z)$ if at each point (x, y, z) of space Ω, \vec{F} is tangent to the curve in F passing through the point

(2) Physical problems

(a)Velocity field of a fluid
streamlines , flow lines

(b)magnetic field
lines of forces

(3) Streamline equation

Given a vector field $\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$

Assume that curve c is streamline of \vec{F} and has position vector as

$$\vec{R}(\zeta) = x(\zeta) \vec{i} + y(\zeta) \vec{j} + z(\zeta) \vec{k}$$

Tangent vector to c is

$$\vec{R}'(\zeta) = x'(\zeta) \vec{i} + y'(\zeta) \vec{j} + z'(\zeta) \vec{k}$$

$\because c$ is a streamline of \vec{F} , so $\vec{F}(x(\zeta), y(\zeta), z(\zeta))$ also tangent to c

$$\therefore \vec{F}(x(\zeta), y(\zeta), z(\zeta)) \parallel \vec{R}'(\zeta)$$

$$\Rightarrow \vec{R}'(\zeta) = t \vec{F}(x(\zeta), y(\zeta), z(\zeta))$$

$$\therefore \begin{cases} x'(\zeta) = tf_1 \\ y'(\zeta) = tf_2 \\ z'(\zeta) = tf_3 \end{cases} \text{ A system of O.D.E for coordinate functions of streamlines}$$

If $f_1, f_2, f_3 \neq 0$ then

$$\begin{cases} \frac{dx}{f_1} = td\zeta \\ \frac{dy}{f_2} = td\zeta \\ \frac{dz}{f_3} = td\zeta \end{cases} \Rightarrow \frac{dx}{f_1} = \frac{dy}{f_2} = \frac{dz}{f_3}$$

EX: Given: Vector Field

Find: Streamlines of

Sol: Streamline equation

$$\frac{dx}{x^2} = \frac{dy}{2y} = \frac{dz}{-1}$$

∴ parametric equation of streamlines

$$(1) \frac{dx}{x^2} = \frac{dz}{-1}$$

$$\left\{ \begin{array}{l} x = \frac{1}{z-k} \\ y = ae^{-2z} \\ z = z \end{array} \right.$$

$$\int \frac{dx}{x^2} = \int \frac{dz}{-1} + k$$

$$\Rightarrow \frac{-1}{x} = -z + k$$

$$\therefore x = \frac{1}{z-k} \quad \dots\dots\dots \textcircled{1}$$

$$\frac{dy}{2y} = \frac{dz}{-1}$$

$$(2) \int \frac{dy}{2y} = \int \frac{dz}{-1} + c \quad \dots\dots\dots \textcircled{2}$$

$$\frac{1}{2} \ln y = -z + c$$

$$\Rightarrow y = ae^{-2z}$$

11.4 The Gradient Field 梯度場

1. Scalar field $\varphi(x, y, z)$

A real-valued function EX: Temperature T(X,Y,Z) ; Pressure P(X,Y,Z)

2. Gradient of a scalar field

$\text{grad } (\varphi)$ or $\nabla \varphi$

$$\nabla \varphi = \frac{\partial \varphi}{\partial x} \vec{i} + \frac{\partial \varphi}{\partial y} \vec{j} + \frac{\partial \varphi}{\partial z} \vec{k}$$

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$$

= del operator

scalar field $\xrightarrow{\nabla}$ vector field

3. Directional Derivative

(1) Definition

Directional Derivative of a scalar field at $P_0(x_0, y_0, z_0)$

In the direction of unit vector $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ is given by

$$\begin{aligned} D\vec{u}\varphi(P_0) &= \frac{d}{dt}\varphi(x_0 + at, y_0 + bt, z_0 + ct) \Big|_{t=0} \\ &= \nabla\varphi(P_0) \cdot \vec{u} \end{aligned}$$

(2) Proof of $D\vec{u}\varphi(P_0) = \nabla\varphi(P_0) \cdot \vec{u}$

By chain rule

$$\begin{aligned} \frac{d}{dt}\varphi(x_0 + at, y_0 + bt, z_0 + ct) &= \frac{\partial\varphi}{\partial x} \cdot \frac{\partial(x_0 + at)}{\partial t} + \frac{\partial\varphi}{\partial y} \cdot \frac{\partial(y_0 + bt)}{\partial t} + \frac{\partial\varphi}{\partial z} \cdot \frac{\partial(z_0 + ct)}{\partial t} \\ &= \frac{\partial\varphi}{\partial x} a + \frac{\partial\varphi}{\partial y} b + \frac{\partial\varphi}{\partial z} c \\ &= \nabla\varphi \cdot \vec{u} \end{aligned}$$

$$\therefore t = 0$$

$$\begin{aligned} \frac{d}{dt}\varphi(x_0 + at, y_0 + bt, z_0 + ct) \Big|_{t=0} &= D\vec{u}\varphi(P_0) \\ &= \nabla\varphi(P_0) \cdot \vec{u} \end{aligned}$$

EX: Given: scalar field $\varphi(x, y, z) = x^2 y - xe^z$

Find: Directional derivative of φ at point(2,-1,0)

In the direction of $2\vec{i} - 4\vec{j} + \vec{k}$

Sol: Gradient of φ

$$\begin{aligned} \nabla\varphi &= \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k} \\ &= (2xy - e^z)\vec{i} + x^2\vec{j} - xe^z\vec{k} \\ \therefore \nabla\varphi(2, -1, 0) &= -5\vec{i} + 4\vec{j} - 2\vec{k} \end{aligned}$$

$$\vec{u} = \frac{(2\vec{i} - 4\vec{j} + \vec{k})}{\sqrt{2^2 + (-4)^2 + 1^2}} = \frac{1}{\sqrt{21}}(2\vec{i} - 4\vec{j} + \vec{k})$$

$$\therefore D\vec{u}\varphi(2, -1, 0) = \nabla\varphi(2, -1, 0)\vec{u} = \frac{-28}{\sqrt{21}}$$

4. Level surface

A level surface is a locus of points(X,Y,Z) satisfying
 $\varphi(x, y, z) = \text{Constant}$ of a scalar function $\varphi(x, y, z)$

EX: A scalar field $\varphi(x, y, z) = x^2 + y^2 + z^2$

Sol: Level surface of $\varphi(x, y, z) = x^2 + y^2 + z^2$ is

$$x^2 + y^2 + z^2 = k = \begin{cases} \text{a sphere surface } k > 0 \\ \text{a single point } k = 0 \end{cases}$$

5. Tangent Plane

(1) Definition

A plane is determined by tangent vectors at P_0 on the level surface.

(2) Normal vector of a tangent plane

A vector normal to the tangent plane at P_0 is given by $\nabla\varphi(P_0)$ which is the gradient of φ at P_0

Suppose smooth curve C on $\varphi = k$ passing P_0 , thus the parametric

equations of c are $x = x(t)$, $y = y(t)$, $z = z(t)$

$\therefore \varphi(x(t), y(t), z(t))$

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial\varphi}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial\varphi}{\partial z} \frac{\partial z}{\partial t} = 0$$

$$\Rightarrow \frac{\partial\varphi}{\partial x} x'(t) + \frac{\partial\varphi}{\partial y} y'(t) + \frac{\partial\varphi}{\partial z} z'(t) = 0$$

$$\Rightarrow \left(\frac{\partial\varphi}{\partial x} \vec{i} + \frac{\partial\varphi}{\partial y} \vec{j} + \frac{\partial\varphi}{\partial z} \vec{k} \right) \bullet (x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}) = 0$$

$$\Rightarrow \nabla\varphi(x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}) = 0$$

\therefore at P_0

$$\nabla\varphi(P_0) \bullet (x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}) = 0$$

$\Rightarrow \nabla\varphi(P_0)$ orthogonal to every tangent vector at P_0

$\Rightarrow \nabla\varphi(P_0)$ is the normal vector of the tangent plane to the level surface at P_0

(3) Equation of tangent plane at P_0

$$\nabla\varphi(P_0) \bullet [(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}] = 0$$

$$\text{Proof: } P_0 \bar{P} = (x, y, z) - (x_0, y_0, z_0) = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$$

$\Rightarrow \nabla \varphi(P_0)$ is normal vector of tangent plane of P_0

$\therefore \nabla \varphi(P_0)$ orthogonal to $P_0 \bar{P}$

$\Rightarrow \nabla \varphi(P_0) \bullet P_0 \bar{P} = 0$

Ex. Give: $\varphi(x, y, z) = z - \sqrt{x^2 + y^2}$, level surface $\varphi(x, y, z) = 0$

Find: A normal vector and tangent plane at $(1, 1, \sqrt{2})$

$$\text{Sol: } \nabla \varphi = \frac{\partial \varphi}{\partial x} \vec{i} + \frac{\partial \varphi}{\partial y} \vec{j} + \frac{\partial \varphi}{\partial z} \vec{k} = -\frac{x}{\sqrt{x^2 + y^2}} \vec{i} - \frac{y}{\sqrt{x^2 + y^2}} \vec{j} + \vec{k}$$

$$\varphi = 0 \Rightarrow z = \sqrt{x^2 + y^2}$$

$$\therefore \nabla \varphi = -\frac{x}{z} \vec{i} - \frac{y}{z} \vec{j} + \vec{k}, z \neq 0$$

\therefore normal vector of level surface $\varphi = 0$

at $(1, 1, \sqrt{2})$ is

$$\nabla \varphi(1, 1, \sqrt{2}) = -\frac{1}{\sqrt{2}} \vec{i} - \frac{1}{\sqrt{2}} \vec{j} + \vec{k}$$

tangent plane at $(1, 1, \sqrt{2})$

$$\nabla \varphi(1, 1, \sqrt{2}) \bullet [(x-1)\vec{i} + (y-1)\vec{j} + (z-\sqrt{2})\vec{k}] = 0$$

or

$$x + y - \sqrt{2}z = 0$$

11.5 Divergence and Curl

1. Divergence

$$\begin{aligned} \text{div } \vec{F} &= \nabla \bullet \vec{F} \\ &= \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \bullet (f\vec{i} + g\vec{j} + h\vec{k}) \\ &= \frac{\partial f}{\partial x} \vec{i} \cdot \vec{i} + \frac{\partial g}{\partial y} \vec{j} \cdot \vec{j} + \frac{\partial h}{\partial z} \vec{k} \cdot \vec{k} \\ &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \end{aligned}$$

vector field $\xrightarrow{\text{div}}$ scalar field

$$\text{Ex. } \vec{F} = 2xy\vec{i} + xyz^2\vec{j} + ze^{x+y}\vec{k} \quad \text{div } \vec{F} = ?$$

Sol: $\text{div } \vec{F} = \nabla \cdot \vec{F}$

$$\begin{aligned} &= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(ze^{x+y}) \\ &= 2y + xz^2 + e^{x+y} \end{aligned}$$

2.Curl

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \vec{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \vec{k}$$

Vector field $\xrightarrow{\text{curl}}$ vector field

$$\text{Ex. } \vec{F} = y\vec{i} + 2xz\vec{j} + ze^x\vec{k} \quad \text{Curl } \vec{F} = ?$$

$$\text{Sol: curl } \vec{F} = \nabla \times \vec{F} = -2x\vec{i} - ze^x\vec{j} + (2z-1)\vec{k}$$

3. Properties b/w gradient,divergence and curl

(1) Curl of a gradient

$$\text{Curl}(\nabla \varphi) = \nabla \times (\nabla \varphi) = 0$$

$$\text{Proof: } \nabla \times (\nabla \varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 \varphi}{\partial z \partial y} - \frac{\partial^2 \varphi}{\partial y \partial z} \right) \vec{i} + \left(\frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \varphi}{\partial z \partial x} \right) \vec{j} + \left(\frac{\partial^2 \varphi}{\partial y \partial x} - \frac{\partial^2 \varphi}{\partial x \partial y} \right) \vec{k} = 0$$

(2) Divergence of a curl

$$\text{Div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

(3) Others

$$<1> \quad \nabla(\vec{F} \bullet \vec{G}) = (\vec{F} \bullet \nabla)\vec{G} + (\vec{G} \bullet \nabla)\vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F})$$

$$\vec{F} \bullet \nabla = f_1 \frac{\partial}{\partial x} \vec{i} + f_2 \frac{\partial}{\partial y} \vec{j} + f_3 \frac{\partial}{\partial z} \vec{k}$$

$$<2> \quad \nabla \bullet (\vec{F} \times \vec{G}) = \vec{G} \bullet (\nabla \times \vec{F}) - \vec{F} \bullet (\nabla \times \vec{G})$$

$$<3> \quad \nabla \times (\vec{F} \times \vec{G}) = \nabla(\vec{F} \bullet \vec{G}) - \nabla^2 \vec{F}$$

$$\nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}$$

$$\begin{aligned} <4> \quad \nabla \times (\bar{F} \times \bar{G}) &= (\bar{G} \bullet \nabla) \bar{F} - (\bar{F} \bullet \nabla) \bar{G} - (\nabla \bullet \bar{F}) \bar{G} \\ \bar{G} \bullet \nabla &= g_1 \frac{\partial}{\partial x} \vec{i} + g_2 \frac{\partial}{\partial y} \vec{j} + g_3 \frac{\partial}{\partial z} \vec{k} \end{aligned}$$