CHAP.1 First-order Differential Equation

1.1 preliminary Concepts

(1) Definition and terminology

Differential equation is an equation containing the derivatives or differentials of one or more dependent variables with respect to one or more independent variables.

General Form

$$F(x, y, z, t, u, u_x, u_y, ...v, v_x, ...) = 0$$

Ex.

$$y''(x) + y(x) = 4\sin(3x)$$

$$\frac{d^4 w}{dt^4} - (w(t))^2 = e^{-t}$$

Type of Differential Equation

A. Ordinary Differential Equation (O.D.E)

D.E. with one independent variable.

$$F(x, u', u'', ..., u^{(n)}) = 0$$

Ex.

$$(1) y' - y'' - e^y = 0$$

$$(2) y' - 2 = 0$$

$$(3) y'' - 2y' + 6y = 0$$

$$(4) (x + y)dx - 4ydy = 0$$

B. Partial Differential Equation <P.D.E>

D.E. with two or more independent variables

$$F(x, y, u, v, u_x, v_y, ...) = 0$$

Ex.

(1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (一個因變數,兩個自變數)

(2) $\frac{\partial u}{\partial x} = \frac{-\partial v}{\partial y}$ (兩個因變數,兩個自變數)

(2) Order and Degree

A. Order(階): The order of highest derivative in D.E. is called the order of the equation.

Ex.

(1) $\frac{d^4 w}{dt^2} - (w(t))^2 = e^{-T}$ (4^{th} order O.D.E)

(2) $u\left(\frac{\partial^3 u}{\partial x^2 \partial y}\right)^2 + \frac{\partial^2 u}{\partial x \partial y} = 0$ (3^{rd} P.D.E)

B. Degree: The degree of a D.E is the highest power of the highest derivatives.

Ex.

(1) $u(\frac{\partial^3 u}{\partial x^2 \partial y})^2 + \frac{\partial^2 u}{\partial x \partial y} = 0$ (2^{nd} degree)

(2) $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + (\frac{\partial u}{\partial x})^2$ (2^{nd} degree (1^{st} order) P.D.E)

(3) Linearity.

A. Linear

A D.E is said to be linear if it is linear in the dependent variable and all its derivatives with coefficients depending only on the independent variables.

Ex.

(1) $y''+4y' = 3Sin(x)$ linear O.D.E

(2) $y\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$ linear P.D.E

C. Nonlinear

The D.E., which is not linear, is called a nonlinear equation.

Ex：d 4 w/dt 4 -(w(t)) 2 =e <nonlinear>

x・(dy/dx)+(dy/dx) 2 =y <nonlinear>

y・y″ -2y ˊ =x <nonlinear>

(4) Homogenity

A. Homogeneous D.E.

AUxx+BUxy+CUyy=D(x,y)≡0

Ex. $y' - y^2 = 0$

B. Nonhomogeneous D.E.

AUxx+BUxy+CUyy=D(x,y)≠0

Ex. $y' - y^2 = e \neq 0$

(5) Solution of a D.E.

Any function φ defined on some interval which, when substituted into a D.E., reduced the equation to an identity is said to be a solution of the equation on the interval.

Ex：D.E. F(x, y, y')=0

Let $y=\varphi(x)$

If $F(x, \varphi(x), \varphi'(x))=0$ for all x in I

Then $\varphi(x)$ is a solution of F(x, y, y')=0 on the interval I.

Ex： D.E. $dy/dx - xy^{0.5}=0$ (1^{st} order, 1^{st} degree, nonlinear, homogeneous, O.D.E.)

Let $y=(x^4)/16$ $(dy/dx)=x^3/4$

$y^{0.5}=x^2/4$ ∴ $x^3/4-xx^2/4=0$

∴ $y=x^4/16$ is a solution of $dy/dx - xy^{0.5}=0$

A. General solution and Particular solutions

A general solution of a D.E. is a family of solutions containing independent arbitrary constants (or essential parameters)

Ex：D.E y'+y=2

$\phi(x)=2+ke^{-x}$ is a general solution of y'+y=2 for all x and any number k.

A particular solution is a solution can be obtained from the general solution

k=1 $f(x)=2+e^{-x}$

k=-1 $g(x)=2-e^{-x}$

are particular solutions of y'+y=2.

B. Explicit solution and Implicit solution

Solutions of a D.E. can be distinguished as explicit or implicit solutions, depending on solutions which are expressed in explicit function or implicit function．

Ex. D.E. $y'+y=2$

Solution $y=2+k\,e^{-x}$ ----------------------explicit solution

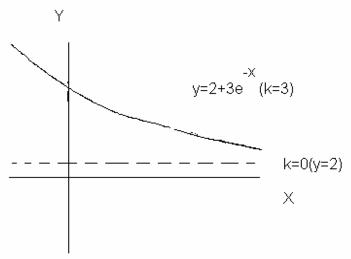
$(y-2)e^{x}=k$ ----------------------implicit (f(x，y)=c)

(6) Integral Curves

Graphs of solution of 1st order ordinary differential equation

Ex: O.D.E $y'+y=2$

General solution $y=2+ke^{(-x)}$



(7) Initial-value problems

1st order D.E. $F(x,y,y')=0$

initial condition $y(X_0) = Y_0$

Ex: O.D.E $\underline{y+y'=2 \quad y(1)= -5}$ solution =?

Sol: general solution $y=2+ke^{(-x)}$

$y(1) = 2+ke^{-1}= -5$

$k= -7e$

∴ solution of the given I .V. P is

$y(x) = 2-7e\,e^{(-x)}$

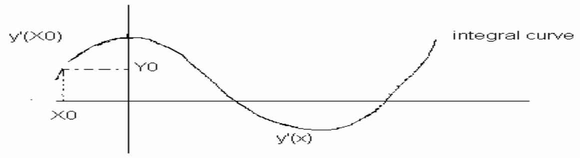
$= 2-7e^{(-x+1)}$

A particular solution (a special integral curve) with k = -7e .

(8) Direction fields

A method derives an approximate solution (Direction field) of differential equation $y' = f(x ,y)$ by using sketches of short line segments of slope f(x, y) drawn at selected points (x, y).

Integral curve through point (x, y) has a slope f(x, y).



Ex. y’+y=2

y’ = 2-y =f(x,y)   Direction fields ?

1.2 Separable Equations (可分離方程式)

1. Definition

A D.E is called separable if it can be written in the following form:

(1) $y' = A(x)B(y)$

(2) $dy/B(y) = A(x)dx \quad , \quad B(y) \neq 0$

(3) $dy/dx = g(x)/h(y) \quad , h(y) \neq 0$

(4) $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$

2. Method of solution

$dy/B(y) = A(x)dx$

$\therefore \int dy/B(y) = \int A(x)dx + C$

Ex. $y' + y = 2$, solution=?

sol: $y' = (2-y)1$ →Separable D.E

$dy/(2-y) = dx \quad ,(y \neq 2)$

$\therefore \int dy/(2-y) = \int dx + c$

$-\ln | y-2 | = x + c$

$| y-2 | = e^{-(x+c)} = ke^{-x}$

$y-2 = \pm ke^{-x}$

Ex. Initial value problem

$y'=y^2e^{-x}$, $y(1)=4$

sol: $y'=y^2e^{-x}=B(y)A(x)$

∴separable

$\rightarrow dy/y^2=e^{-x}dx$,$(y\neq 0)$

$\int dy/y^2=\int e^{-x}dx + c$

∴ $-1/y = -e^{-x}+c$

∴ $y(x)=1/e^{-x}-c$, c=?

$y(1)=1/e^{-1}-c=4$, $c=e^{-1}-1/4$

∴ $y(x)=1/e^{-x}-e^{-1}+1/4$

3. Some applications of separable D.E

(1) Newton's Law of Cooling (牛頓冷卻定律)

$dT/dt=k(T-T_0)$, T_0: room temperature 常溫

(2) Carbon dating (c^{14})

$dm/dt=km(t)$, 利用 c^{14} 衰弱程度求化石距今?年

(3) Torricelli's Law

$dv/dt=-kA\boldsymbol{v}$,

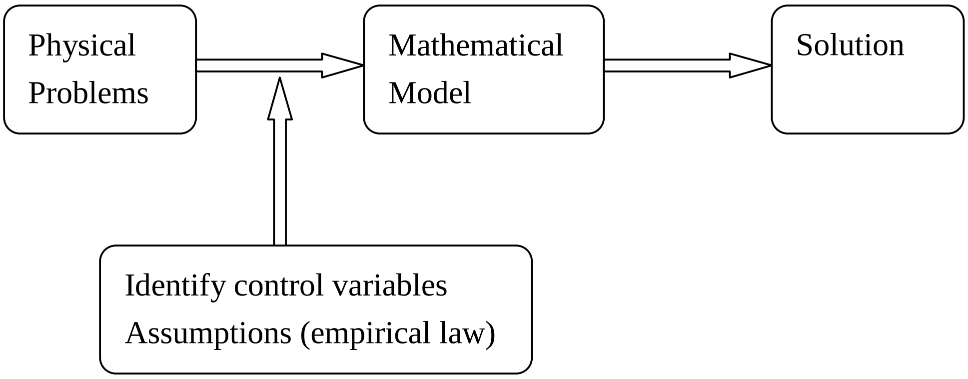
v:體積, A:孔面積, ***v***:液體流速

應用:一桶液體下有一孔,計多久流光

(4) Propulation/bateria growth

$dp/dt=kp$

4. Mathematical Model



1.3 Linear differential equations

1. Definition:

A 1^{st}-order D.E is linear if it has the form:

$y'(x)+p(x)y(x)=q(x)$,

$p(x),q(x)$ are continuous.

2. Method of solution:

Multiplying both sides of the equation by $e^{\int p(x)dx}$ to yield

$e^{\int p(x)dx}y'+e^{\int p(x)dx}p(x)y(x)=e^{\int p(x)dx}q(x)$

$(e^{\int p(x)dx}y)'=e^{\int p(x)dx}q(x)$

$d(e^{\int p(x)dx}y)'=e^{\int p(x)dx}q(x)dx$

$\int d(e^{\int p(x)dx}y)= \int e^{\int p(x)dx}q(x)dx + c$

$\therefore\ y(x)=e^{-\int p(x)}(\int e^{\int p(x)dx}q(x)dx) + ce^{-\int p(x)dx}$

Ex1. $y'+y=2$

$p(x)=1$, $q(x)=2$ linear D.E

$\therefore e^{\int p(x)dx}=e^{\int 1dx}=e^{x}$

multiply both sides of $y'+y=2$ by e^x to obtain

$e^xy'+e^xy=2e^x$

$(e^xy)'=2e^x$

$e^xy= \int 2e^xdx+c=2e^x+c$

$\therefore y=2+ce^{-x}$

Ex2. $y'+xy=2$

$p(x)=x$, $q(x)=2$ linear D.E.

integration factor $e^{\int p(x)dx}=e^{\int xdx}=e^{x2}$

$e^{x2}(y'+xy)=2e^{x2}$

$(e^{x2}y)'=2e^{x2}$

$e^{x2}y=\int 2e^{x2}dx+c$

$y(x)=e^{-x2}\underline{\int 2e^{x2}dx}+ce^{-x2}$ #

1.4 EXACT DIFFERENTIAL EQUATIONS

1.DEFINITION

If D.E M(x,y)dx + N(x,y)dy = 0 or M(x,y) + N(x,y)y' = 0 can be expressed in terms of total differential of function $\phi(x,y)$, then the D.E is said to be exact , and function $\phi(x,y)$ is called a potential function for the D.E,

$\frac{\partial \phi}{\partial x} =$ M(x,y) , $\frac{\partial \phi}{\partial y} = N$ (x,y).

d$\phi(x,y)$ = $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}$ dy

= M(x,y)dx + N(x,y)dy

$\therefore$ $\frac{\partial \phi}{\partial x} =$ M(x,y) , $\frac{\partial \phi}{\partial y} = N$ (x,y)

EX. $x^2y^3dx + x^3y^2dy = 0$, exact ?

Sol : $\phi(x,y)$ = $\frac{1}{3}x^3y^3$

d$\phi(x,y)$ = d($\frac{1}{3}x^3y^3$)

= $y^3d(\frac{1}{3}x^3) + \frac{1}{3}x^3d(y^3)$

= $x^2y^3dx + x^3y^2dy = 0$

$\therefore$ D.E is exact .

2. CRITERION FOR AN EXACT D.E

Suppose M(x,y) , N(x,y) , $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ are continuous for all (x,y) within a rectangle R in the plane . Then M(x,y) + N(x,y)y' = 0 is exact on R iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for each (x,y) in R.

(A) NECESSITY :

If the expression M(x,y) + N(x,y)y' is exact , then there exists some function φ (x,y) such that

M(x,y)dx + N(x,y)dy = d φ (x,y) = $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$

$$\frac{\partial \phi}{\partial x} = M \quad , \quad \frac{\partial \phi}{\partial y} = N$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} \qquad \rightarrow \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(B) SUFFICIENCE

Show that there exist a function φ (x,y) for which $\frac{\partial \phi}{\partial x} = M$ (x,y) and $\frac{\partial \phi}{\partial y} = N$ (x,y), whenever $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$\varphi(x,y) = \int_{x_0}^{x} M(\xi, y_0)d\xi + \int_{y_0}^{y} N(x,\eta)d\eta$$

$$\therefore \quad \frac{\partial \phi}{\partial y} = 0 + N(x,y)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} M(\xi, y_0)d\xi + \frac{\partial}{\partial x}\int_{y_0}^{y} N(x,y)d\eta$$

$$= M(x, y_0) + \int_{y_0}^{y} \frac{\partial N(x,\eta)}{\partial x} d\eta \qquad \left(\frac{\partial M(x,y)}{\partial y} = \frac{\partial N}{\partial x}\right)$$

$$= M(x, y_0) + \int_{y_0}^{y} \frac{\partial M(x,\eta)}{\partial \eta} d\eta$$

$$= M(x, y_0) + M(x,y)\Big|_{y_0}^{y} = M(x,y)$$

3. METHOD OF SOLUTION

D.E M(x,y)dx + N(x,y)dy = 0 or M(x,y)+ N(x,y)y' = 0

(1) show that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(2) φ (x,y) exists

$$d\varphi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = Mdx + Ndy$$

$$\frac{\partial \phi}{\partial x} = M(x,y)$$

$$\therefore \quad \varphi(x,y) = \int M(x,y)dx + g(y)$$

(3) $\frac{\partial \phi}{\partial y} = N$

$\therefore$ $\frac{\partial}{\partial y}\int M(x,y)dx$ + g' (y) = N

$\therefore$ g' (y) = N - $\frac{\partial}{\partial y}\int M(x,y)dx$

(4) Obtain g(y) by integration with respect to y.

Other method :

(1) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(2) $\frac{\partial \phi}{\partial x} = M(x,y)$

$\phi(x,y) = \int M(x,y)dx + g(y)$

(3) $\frac{\partial \phi}{\partial y} = N(x,y)$

$\phi(x,y) = \int N(x,y)dy + h(x)$

(4) obtain $\phi(x,y)$ by comparing the term of $\phi(x,y)$ in (2) and (3)

EX. Solve $x^2 + 3xy + (4xy + 2x)$ y' = 0

(1) y' = - $\frac{x+3y}{4y+2}$ $\neq$ A(x)B(y)

$\therefore$ not separable

(2) y' + $\frac{x^2+3xy}{4xy+2x}$ = 0

(4y+2) y' + 3y + x = 0

$\therefore$ nonlinear D.E

(3) M(x,y) = $x^2 + 3xy$

N(x,y) = 4xy + 2x

$\frac{\partial M}{\partial y} = 3x$, $\frac{\partial N}{\partial x} = 4y + 2$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \therefore$ not exact

EX. Solve $2xydx + (\chi^2 - 1)dy = 0$

M(x,y) = 2xy

N(x,y) = $\chi^2 - 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x \qquad \therefore$ exact D.E

$\frac{\partial \phi}{\partial x}$ = M(x,y) = 2xy

$\frac{\partial \phi}{\partial y}$ = N(x,y) = $\chi^2 - 1$

φ (x,y) = $\int M dx + g(y)$

= $\int 2xydx + g(y)$

= $\chi^2 y + g(y)$

$\frac{\partial\left(\chi^2 y + g(y)\right)}{\partial y} = \chi^2 +$ g' (y) = N = $\chi^2 - 1$

$\therefore$ g' (y) = -1

g(y) = -y+c

φ (x,y) = $\chi^2 y - y = C$ ($\because$ dφ = dC = Mdx + Ndy = 0)

other method :

N(x,y) = $\chi^2 - 1 \quad \rightarrow \quad \varphi$ (x,y) = $\int Ndy +$ h(x)

= $\chi^2 y - y +$ h(x)

= $\chi^2 y +$ g(y)

$\therefore$ g(y) = -y , h(x) = 0

$\rightarrow \quad \varphi$ (x,y) = $\chi^2 y - y = C$

1.5 Integrating factors

1. Definition

A function μ (x,y) is an integrating factor for M(x,y)+N(x,y)y'=0 if μ (x,y) $\neq 0$ for all (x,y) in R , and μ M+ μ Ny'=0 is exact on R .

M(x,y)+N(x,y)y'=0 is not exact !

Multiplied by μ (x,y) , then

μ (x,y)M(x,y)+ μ (x,y)N(x,y)y'=0 is exact

$$\therefore \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \text{or} \quad M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial X} + \mu\frac{\partial N}{\partial x}$$

μ (x,y)=?

2. Method of finding μ (x,y)

(1) μ (x,y)= μ (x) only

$$\text{then} \quad \mu\frac{\partial M}{\partial Y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

$$\mu(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial \mu}{\partial x}$$

$$\frac{(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})}{N} = \frac{\frac{d\mu(x)}{dx}}{\mu} \quad \text{or} \quad \frac{(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})}{N}dx = \frac{d\mu}{\mu}$$

$$if \ \frac{(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})}{N} = F(x) \quad \text{Function of x only !}$$

then F(x)dx= $\frac{d\mu}{\mu}$, $\int F(x)dx = \int \frac{d\mu}{\mu}$, $\ln \mu = \int F(x)dx$

hence, I.F. μ (x)=e$^{\int F(x)dx}$

Ex. Solve (x-xy)-y'=0

Sol. Separable and linear

M(x,y)=x-xy

N(x,y)=-1

$$\frac{\partial M}{\partial y} = -x \neq \frac{\partial N}{\partial x} = 0$$ not exact D.E.

Find an integrating factor μ (x,y) to make μ (x'-xy-y')=0 to be exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-x-0}{-1} = x$$ function of x

$\therefore \mu$ (x.y)= μ (x)=e$^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$ =e$^{\int xdx}$ =e$^{\frac{x^2}{2}}$

mutiply (x-xy)-y'=0 by e$^{\frac{x^2}{2}}$ to yield

$$\left.\begin{aligned} & e^{\frac{x^2}{2}}(x-xy) - e^{\frac{x^2}{2}}y' = 0 \\ & \frac{\partial[e^{\frac{x^2}{2}}(x-xy)]}{\partial y} = -xe^{\frac{x^2}{2}} \\ & \frac{\partial(-e^{\frac{x^2}{2}})}{\partial x} = -xe^{\frac{x^2}{2}} \end{aligned}\right\} Exact$$

$$\frac{\partial \varphi}{\partial x} = e^{\frac{x^2}{2}}(x-xy)$$

$$\frac{\partial \varphi}{\partial y} = -e^{\frac{x^2}{2}}$$

$$\varphi\,(\mathrm{x,y}) = -\int e^{\frac{x^2}{2}} dy + h(x) = -e^{\frac{x^2}{2}} y + h(x)$$

$$\frac{\partial \varphi}{\partial x} = -xe^{\frac{x^2}{2}} y + h'(x) = e^{\frac{x^2}{2}} (x - xy)$$

$$h'(x) = xe^{\frac{x^2}{2}} \Rightarrow h'(x) = e^{\frac{x^2}{2}}$$

$$\therefore \varphi(x, y) = -e^{\frac{x^2}{2}} y + e^{\frac{x^2}{2}} = C$$

(2) $\mu(x, y) = \mu(y)$ only

$$\therefore M \frac{\partial \mu}{\partial y} + \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x}$$

$$M \frac{\partial \mu}{\partial y} = \mu(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$$

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy = \frac{d\mu}{\mu}$$

$$if \; \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \quad \text{only}$$

then integrating factor $\mu(x, y) = \mu(y) = e^{\int g(y)dy}$

Ex. $\frac{2xy}{y-1} - y' = 0$

Sol. separable , nonlinear, not exact

2xy-(y-1)y'=0

$$\frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = 0$$

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{0-2x}{2xy} = -\frac{1}{y} = g(y)$$

$$\mu(x,y) = \mu(y) = e^{\int -\frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

(3) $\mu(x,y) = \mu(xy)$

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x} + M\frac{\partial \mu}{\partial y}$$

$$\frac{\partial \mu}{\partial x} = \frac{d\mu}{d(xy)} \bullet \frac{d(xy)}{dx} = y\frac{d\mu(xy)}{d(xy)}$$

$$\frac{\partial \mu}{\partial y} = \frac{d\mu}{d(xy)} \bullet \frac{d(xy)}{dy} = x\frac{d\mu(xy)}{d(xy)}$$

$$\therefore \mu(xy)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = (Ny - Mx)\frac{d\mu(xy)}{d(xy)}$$

$$\frac{\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)}{Ny - Mx}d(xy) = \frac{d\mu(xy)}{d(xy)}$$

$$if \ \frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{Ny-Mx}=h(xy)\,\text{only}$$

$$\therefore \mu(x,y)=\mu(xy)=e^{\int h(xy)d(xy)}$$

Ex. $2y^2-9xy+(3xy-6x^2)y'=0$

Sol. $\frac{\partial M}{\partial y}=4y-9x\neq\frac{\partial N}{\partial x}=3y-12x$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{Ny-Mx}=\frac{(4y-9x)-(3y-12x)}{(3xy-6x^2)y-(2y^2-9xy)x}$$

$$=\frac{y+3x}{xy^2+3x^2y}=\frac{y+3x}{xy(y+3x)}$$

$$=\frac{1}{xy}=h(xy)$$

$\therefore$ integrating factor

$$\therefore \quad \mu(x,y)=\mu(xy)=e^{\int\frac{1}{xy}d(xy)}=e^{\ln xy}=xy$$

(4) $\mu(x,y)=\mu(x+y)$ only

$$\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\mu=N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}$$

$$\frac{\partial\mu(x+y)}{\partial x}=\frac{d\mu(x+y)}{d(x+y)}\bullet\frac{d(x+y)}{dx}=\frac{d\mu(x+y)}{d(x+y)}$$

$$\frac{\partial\mu(x+y)}{\partial y}=\frac{d\mu(x+y)}{d(x+y)}$$

$$\therefore \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \mu (N - M) = \frac{d\mu(x+y)}{d(x+y)}$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N - M} d(x+y) = \frac{d\mu(x+y)}{d(x+y)}$$

$$if \ \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N - M} = \phi(x+y) \quad \text{only}$$

then $\mu(x, y) = \mu(x+y) = e^{\int \phi(x+y) d(x+y)}$

1.6 Homogeneous, Bernoulli, and Riccati Equations

1. Homogeneous Differential Equation

(1) Homogeneous function

If there exists a constant n such that $f(tx,ty)=t^n f(x,y)$ for every number t, then $f(x,y)$ is said to be a homogeneous function of degree n.

EX. $f(x,y)=\sqrt{x^3+y^3}$ Homogeneous fn. ?

(sol) $f(tx,ty)=\sqrt{t^3x^3+t^3y^3}$

$$=t^{3/2}\sqrt{x^3+y^3}$$

$$=t^{3/2}f(x,y)$$

$\therefore$ $f(x,y)=\sqrt{x^3+y^3}$ is homogeneous fn. of degree 3/2 .

(2) Homogeneous Equation

A. D.E $M(x,y)+N(x,y)y'=0$ is said to be a homogeneous eqn. if $M(tx,ty)=t^nM(x,y)$ and $N(tx,ty)=t^nN(x,y)$

B. D.E $y'=f(x,y)$ is said to be a homogeneous eqn. if $y'=f(\frac{y}{x})$.

$f(x,y)$ is a homogeneous function of degree n .

$f(tx,ty)=t^n f(x,y)$

then

$$f(x,y)=x^n f(1,\frac{y}{x})$$

or $$f(x,y)=y^n f(\frac{x}{y},1)$$

EX. $f(x,y)=\sqrt{x^3+y^3}$ is homogeneous fn. of degree 3/2

$$f(x,y)=\sqrt{x^3(1+(\frac{y}{x})^3)}$$

$$=x^{3/2}\sqrt{1+(\frac{y}{x})^3} \quad =x^{3/2}f(\frac{y}{x})$$

$$f(x,y)=\sqrt{y^3((\frac{x}{y})^3+1)}$$

$$=y^{3/2}\sqrt{(\frac{x}{y})^3+1} \quad =y^{3/2}f(\frac{x}{y})$$

(3) Method of Solution

Homogeneous D.E → **Substitution of variables** $y = ux$ or $x = vy$ → **Separable D.E**

EX. $xy' = \dfrac{y^2}{x} + y$

(sol). $\because \quad y' = \dfrac{y^2}{x^2} + \dfrac{y}{x} = f(\dfrac{y}{x})$

$\therefore$ given D.E. is homogeneous D.E.

Let $y = ux$, $u = \dfrac{y}{x}$

$$y' = u + u'x$$

$$\therefore \quad u + u'x = u^2 + u$$

$$\frac{du}{dx}x = u^2$$

$$\frac{du}{u^2} = \frac{dx}{x}$$

$$\int \frac{du}{u^2} = \int \frac{dx}{x}$$

$$-\frac{1}{u} = \ln|x| + c$$

$$-\frac{x}{y} = \ln|x| + c$$

$$y = \frac{-x}{\ln|x| + c}$$

2. Bernoulli's Equation

$y' + P(x)y = R(x)y^{\alpha}$ **,** α: any real number

(a) $\alpha = 0$

$y' + P(x)y = R(x)$ linear D.E.

(b) $\alpha = 1$

$y' + P(x)y = R(x)y$

$y' + [R(x) - P(x)]y = A(x)B(y)$ Separable D.E

(c) $\alpha \neq 1$

$y' + P(x)y = R(x)y^{\alpha}$ Nonlinear D.E

(1) Method of solution

change of variables $v = y^{1-\alpha}$

$$y^{-\alpha} y' + P(x) y^{1-\alpha} = R(x) \qquad y \neq 0$$

Let $v = y^{1-\alpha}$

$$v' = (1-\alpha) y^{-\alpha} y'$$

$$\frac{1}{1-\alpha} v' + P(x) v = R(x)$$

or $v' + (1-\alpha) P(x) v = (1-\alpha) R(x)$ 1st-order linear D.E.

EX. $y' + \frac{y}{x} = xy^2$

(sol.) $P(x) = \frac{1}{x}$, $R(x) = x$, $a = 2 \neq 1$

given eqn. is Bernoulli's D.E. with $a = 2$

Let $v = y^{1-\alpha} = \frac{1}{y}$

$$v' = \frac{1}{x} v = -x$$

integrating factor $\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x}(v' - \frac{1}{x} v) = \frac{1}{x}(-x)$$

$$(\frac{1}{x} v)' = -1$$

$$\frac{1}{x} v = -x + c$$

$$v = -x^2 + cx = \frac{1}{y}$$

$$y = \frac{1}{-x^2 + cx}$$

3. Riccati Differential Equation

$$y' = P(x) y^2 + Q(x) y + R(x)$$

if $P(x) = 0$ linear, otherwise nonlinear

(1) Method of solution

Assume that $S(x)$ is known particular solution then a family of solution

$$y(x) = S(x) + U(x)$$

$$(S(x) + U(x))' = P(x)(S + U)^2 + Q(x)(S + U) + R(x)$$

but $S'(x) = P(x)S^2 + Q(x)S + R(x)$

$$U' = P(x)2SU + P(x)U^2 + Q(x)U$$

$$\boxed{U' - 2SP(x)U - Q(x)U = P(x)U^2}$$

Bernoulli's D.E. with $a = 2$

Let $z = U^{1-\alpha} = \frac{1}{U}$

$$\therefore \boxed{\frac{dz}{dx} + (2SP(x) - Q(x))z = -P(x)}$$ 1st-order linear D.E.

find $z(x)$

$$U = \frac{1}{z}$$

$$\boxed{y = S(x) + \frac{1}{z}}$$

1.7 Some Applications of First Order D.E.

1. Application to Mechanics

Newton's second law

F=k•d(mv)/dt

=d(mv)/dt (k=1)

=m dv/dt+V dm/dt (變質量問題)

if m=Constant

then F=m dv/dt =ma

(1) Terminal velacity

αv^2 ↑ ● m dv/dt

mg↓ .

$F=mg - \alpha v^2$ (I.C. v(0)=0)

=m dv/dt free falling body

D.E. $dv/dt = g - \alpha v^2 /m$ a separable D.E

$dv / (g - \alpha V^2 /m) = dt$

$\int [dV/ (g - \alpha V^2 /m)] = \int dt$

$$\sqrt{\frac{m}{\alpha g}}\tanh^{-1}\left(\sqrt{\frac{\alpha}{mg}}v\right) = t + c$$

So, the velocity v(t) is

$$v(t)=\sqrt{\frac{mg}{\alpha}}\tanh(\sqrt{\frac{\alpha g}{m}}(t+c))$$

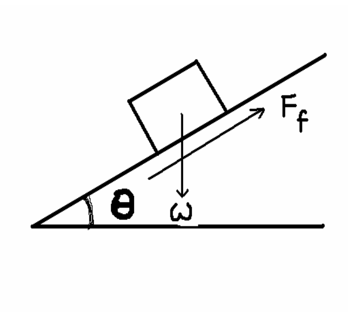
$$v(0)=\sqrt{\frac{mg}{\alpha}}\tanh(\sqrt{\frac{\alpha g}{m}}c)=0$$

c=0

$$v(t)=\sqrt{\frac{mg}{\alpha}}\tanh(\sqrt{\frac{\alpha g}{m}}t)$$

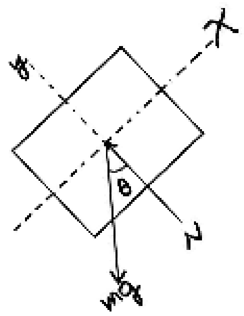
$$\lim_{t\to\infty}v(t)=\lim_{t\to\infty}\sqrt{\frac{mg}{\alpha}}\tanh(\infty)=\sqrt{\frac{mg}{\alpha}}$$ =terminal velocity

(2) Block on an inclined plane



motion of the sliding block??

FBD:



$\sum Fx = ma_x = m\, dv/dt$

$\sum Fy = ma_y$

$\sum Fx = W\sin\Theta - F_f = m\, dv/dt$

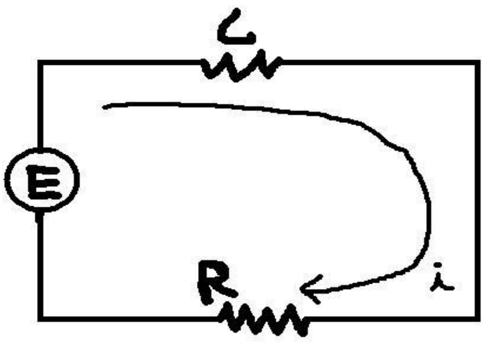
$\sum Fy = -W\cos\Theta + N = 0$

$\therefore$ Equation of motion of the block is

$m(dv/dt) = W\sin\Theta - \mu\, W\cos\Theta$

2. Applications to Electrical

(1) RL series Circuit



$E_L = L\, di/dt$; $E_R = Ri$

Kirchhoff's second law

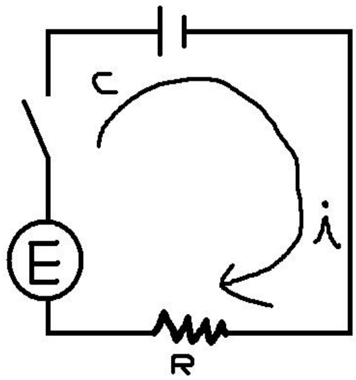
$L\, di/dt + Ri = E(t)$…………First-order liner D.E.

$\therefore$ general solution i(t) is

$i(t) = E/R = Ke^{-RT/L}$ (If E(t) = E =Constant)

$$\lim_{t\to\infty} i(t) = E/R$$

(2) RC series circuit



q/c + Ri = E

i = d q /dt

∴ d q /dt + q/RC = E/R...........First-order D.E.

q(t) = EC(1-e$^{- t/Rc}$)

dq/dt=Ee$^{- t/Rc}$ /R

$$\lim_{t \to 0} i(t) = E/R$$