# Chap 3 The Laplace Transformation

## 3.1 Definition and Basic properties

### 1. Definition of Laplace transform

$₤[f(t)]=\int_0^{\infty} e^{-st} f(t)dt$ is said to be the Laplace transform of function f(t), provided the improper integral converge for all s.

Ex. $f(t) = e^{at}$ $₤[f(t)] = ?$

$$₤[e^{at}] = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = [-e^{-(s-a)t}/(s-a)]_0^{\infty}$$

$$= \lim_{t\to\infty}[-e^{-(s-a)t}/(s-a)]+[1/(s-a)]$$

$$\lim_{t\to\infty}[-e^{-(s-a)t}/(s-a)] \rightarrow 0 \quad \text{as} \quad s-a>0$$

$$\therefore\ ₤[e^{at}] = 1/(s-a) \qquad \text{provided} \quad s > a$$

2. Linearity theorem

$$₤[\alpha f(t) + \beta f(t)] = ₤[\alpha f(t)] + ₤[\beta f(t)] = \alpha ₤[f(t)] + \beta ₤[g(t)]$$

$$= \alpha F(s) + \beta G(s)$$

pf.

$$₤[\alpha f(t) + \beta g(t)] = \int_0^{\infty} e^{-st} [\alpha f(t) + \beta g(t)]dt$$

$$= \alpha \int_0^{\infty} e^{-st} f(t)dt + \beta \int_0^{\infty} e^{-st} g(t)dt$$

$$= \alpha ₤[f(t)] + \beta ₤[g(t)]$$

3. Piecewise continuous

f(t) is piecewise continuous on [a,b] if there are at most a finite number of points $t_k$, k = 1.2……n ($t_{k-1}<t_k$) at which

(1) f(t) has finite jump discontinuity, and $\lim_{t\to t_k} f(t)$ and $\lim_{t\to t_{k-1}} f(t)$ exist.

(2) f(t) is continuous on each internal $(t_{k-1}, t_k)$.

4. Exponential order

   A function f(t) is said to be of exponential order if there exist number b, M>0, and T>0, such that

   $| f(t) | \leqq Me^{bt}$, t>T

5. Existence condition of ₤ [f] (sufficient condition)

   ₤ [f(t)] exists for s > b, if

   (1) f(t) is piecewise continuous on internal [0, k], k>0, and

   (2) there are numbers M and b such that $|f(t)| \leqq Me^{bt}$ for $t \geqq 0$.

   (Exponential order)

pf. Prove ₤ $[f(t)]= \int_0^\infty e^{-st} f(t)dt= \lim_{k\to\infty} \int_0^k e^{-st} f(t)dt$

From condition (1), we have $\int_0^k e^{-st} f(t)dt$ exists because f(t) is

piecewise continuous on [0, k], so is $e^{-st}f(t)$.

From condition(2), we have $\lim_{k\to\infty} \int_0^k e^{-st} f(t)dt = \int_0^\infty e^{-st} f(t)dt$ exists, since

$|\int_0^\infty e^{-st} f(t)dt| \leqq \int_0^\infty | e^{-st} f(t)|dt \quad ( |f(t)| \leqq Me^{bt})$

$$\leqq M\int_0^\infty e^{-st} e^{bt}dt = M\int_0^\infty e^{(b-s)t} dt =M/(s-b) \text{ if } s>b$$

$\therefore \int_0^\infty Me^{(b-s)t}$ dt converges for s>b

so $\int_0^\infty e^{-st} f(t)dt$ has a finite limit (or converge) for s>b.

Ex. $f(t)=t^{-1/2}$

$\because \lim t^{-1/2} = \infty$

$t \to 0+$

$\therefore t^{-1/2}$ is not piecewise continuous on [0, k]

(cond. (1) not satisfied)

but, £ [f(t)] exists = $\sqrt{\pi/s}$

6. Laplace transform of some basic functions

(1) f(t) = 1

$$£[1] = \int_0^{\infty} e^{-st}(1)dt \;=\; -\,e^{-st}/s\Big]_0^{\infty}$$

$$= \lim_{t\to\infty} (-e^{-st}/s) + 1/s$$

$$= 1/s$$

(2) f(t) = $t^n$

$$£[t^n] = \int_0^{\infty} e^{-st}\, t^n\, dt$$

let u = st , t=u/s , dt = 1/s du .

$$£[t^n] = \int_0^{\infty} e^{-u}\,(u/s)^n\, 1/s\, du = 1/s^{n+1} \int_0^{\infty} e^{-u}\, u^n\, du$$

$$£[t^n] \;=\; \Gamma(n+1)/s^{n+1} \qquad n \neq -1,-2,-3.......$$

When n is positive integer $\Gamma(n+1) = n!$

$$£[t^n] \;=\; n!/s^{n+1}$$

(3) f(t) = $e^{at}$ or $e^{-at}$

$$£[e^{at}] = \int_0^{\infty} e^{-st}\, e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt$$

$= -1/(s-a) e^{-(s-a)t} \Big|_0^\infty = 1/(s-a)$   for s > a

Similanly , £[$e^{-at}$] = 1/(s+a) ,  s > 0

(4) f(t) = cos(kt) or sin(kt)

$e^{ikt}$ = cos(kt)+isin(kt)

£[$e^{ikt}$] = 1/(s-ik) = (s+ik)/($s^2+k^2$)

= s/($s^2+k^2$)+ik/($s^2+k^2$)

£[$e^{ikt}$] = £[cos(kt)] +i £[sin(kt)]

∴ £[cos(kt)] = s/($s^2+k^2$) ,  £[sin(kt)] = k/($s^2+k^2$)

(5) f(t) = cosh(kt) or sinh(kt)

£[cosh(kt)] = [($e^{kt}+e^{-kt}$)/2] = 1/2 £[$e^{kt}$]+1/2 £[$e^{-kt}$]

= 1/2[1/(s-k)]+1/2[1/(s+k)]

= s/$s^2-k^2$

Similanly , £[sinh(kt)] = s/$s^2-k^2$

EX  £[$\sin^2$ t] = $\dfrac{2}{s(s^2+4)}$

7. Inverse Laplace transform

Given a function G(s), a function g(t) such that £[g(t)]=G(s) is called an inverse Laplace transform of G(s), and g(t) = £$^{-1}$[G(s)].

(1) Fundemeatal formula

| g(t) | G(s) |
|---|---|
| 1 | 1/s |
| $t^n$ | $n!\ /s^{n+1}$ |
| $e^{+at}$ | 1/s-a |
| cos(kt) | $s/s^2+k^2$ |
| sin(kt) | $k/s^2+k^2$ |

(2) partial fraction　部份分式

Ex. $£^{-1}$ [1/（s-1）（s+2）（s+4）]

Sol. 1/（s-1）（s+2）（s+4）= A/（s-1）+ B/（s+2）+C/（s+4）

A=1/15, B= -1/6, C=1/10

$£^{-1}$ [G(s)]=1/15 $£^{-1}$ [1/(s-1)]-1/6 $£^{-1}$ [1/(s+2)]+1/10 $£^{-1}$ [1/(s+4)]

$=1/15\ e^{t} - 1/6\ e^{-2t} + 1/10\ e^{-4t}$

8. Applications

(1) convent initial value problem into algebra problem

I.V.P → Laplace transform → algebra problem

Solution of I.V.P ← Invese Laplace ← solution of algebra problem

(2) Solve nonhomogenous O.D.E y''+Ay'+By=f（t）where f（t）is not continuous

Ex. f（t）= u（t）　f（t）= $\delta$（t）

(3) Solve equation where an unknown function appears in an integral

$f(t) = e^{-2t} - 3\int_0^t f(t-\alpha)\, e^{-3\alpha} d\alpha$

3.2 Solution of I.V.P using Laplace Transform

1. Laplace transform of a derivative

£[f'(t)]=SF(s)-f(0) , if

(1) f(t) is continuous on $[0,\infty]$ ,

(2) f'(t) is a piecewise continuous on [0,k] for k>0 , and

(3) $\lim_{t\to\infty} e^{-st} f(t)=0$ , if s>0

p.f

$$£[f'(t)]= \int_0^\infty e^{-st} f'(t)dt = \int_0^\infty e^{-st} df(t)$$

$$= e^{-st} f(t)]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$=sF(s) + \lim_{t\to\infty} e^{-st} f(t)-f(0) = sF(s)-f(0)$$

2. Laplace transform of higher derivatives

$$£[f^n(t)] = s^nF(s) – s^{n-1}f(0) -\ldots\ldots\ldots\ldots- f^{n-1}(0)$$

$$= s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0)$$

if

(1) f , f' , f'' …… , $f^{n-1}$ are continuous on $[0,\infty]$,

(2) $f^n$ is piecewise wntinuous on [0,k] for k>0 , and

(3) $\lim_{t\to\infty} e^{-st} f^j(t)=0$ if s>0 , for j = 0,1,2…………,n-1.

Ex.

n=2 £[f''(t)]= $s^2$F(s)-sf(0)-f'(0)

3. Application to I.V.PS

$y'' + 4y' + 3y = e^t$ ; $y(0)=0$ , $y'(0)=2$

Take Laplace transform to yield

$\mathcal{L}[y''+4y'+3y] = \mathcal{L}[e^t]$

$\mathcal{L}[y''] + 4\mathcal{L}[y'] + 3\mathcal{L}[y] = \mathcal{L}[e^t]$

$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2Y(s) - 2$

$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = sY(s)$

$\therefore$ $s^2Y - 2 + 4sY + 3Y = 1/(s-1)$ (algebraic equation for Y(s))

$(s^2+4s+3)Y = 1/(s-1) + 2$

$\therefore$ $Y(s) = 1/(s-1)(s+1)(s+3) + 2/(s+1)(s+3) = A/(s-1) + B/(s+1) + C/(s+3)$

$A=1/8$ , $B=3/4$ , $C=-7/8$

$\therefore$ $y(t) = \mathcal{L}^{-1}[Y(s)]$

$= 1/8\,\mathcal{L}^{-1}[1/s-1] + 3/4\,\mathcal{L}^{-1}[1/s+1] - 7/8\,\mathcal{L}^{-1}[1/s+3]$

$= 1/8\,e^{t} + 3/4\,e^{-t} - 7/8\,e^{-3t}$ .

3.3 Shifting theorem and the Heaviside Function

1. First shift (translation) theorem (shifting in s-variable)

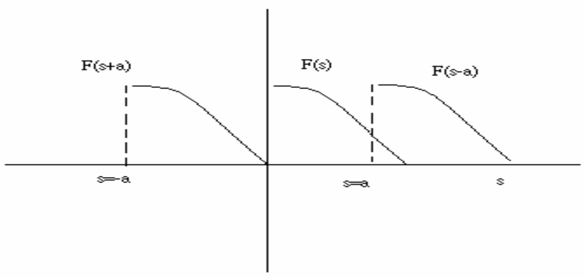
£[ $e^{at}$ f(t)]=F(s-a)  for  s>a+b

if £[f(t)]=F(s) for s>b≧0  and a $\in$ R

proof:

$$£[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a) \quad \text{if} \quad s-a>b$$




Similarly  $£[e^{-at} f(t)] = F(s+a)$

Inverse form

$£^{-1}[F(s \pm a)] = e^{\mp at} f(t)$

if  $£^{-1} F(s) = f(t)$

Ex:  $£[e^{5t} t^3] = ?$

Sol.  $f(t) = t^3$   $e^{at} = e^{5t}$

According to 1st shift theorem we have

$£[e^{at} f(t)] = F(s-a)$   $£[f(t)] = F(s)$

$$\therefore £[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4} = F(s)$$

$$\triangleright \quad £[e^{5t}t^3] = F(s-5) = \frac{6}{(s-5)^4} \qquad (s \to s-5)$$

Ex: $£^{-1}[\frac{s}{s^2+6s+11} = ?$

Sol.

$$\frac{6}{s^2+6s+11} = \frac{s}{(s+3)^2+2} = \frac{s+3-3}{(s+3)^2+2}$$

$$= \frac{s+3}{(s+3)^2+\sqrt{2}^2} - \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(S+3)^2+(\sqrt{2})^2}$$

$$\therefore £^{-1}[\frac{S}{S^2+6S+11}] = £^{-1}[\frac{S+3}{(S+3)^2+(\sqrt{2})^2} - \frac{3}{\sqrt{2}}£^{-1}[\frac{\sqrt{2}}{(S+3)^2+(\sqrt{2})^2}]$$

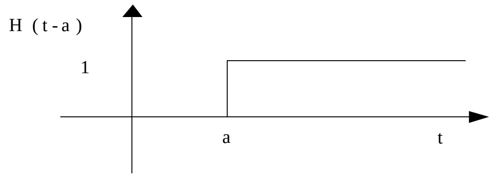
$$= e^{-3t}£^{-1}[\frac{s}{s^2+(\sqrt{2})^2}] - \frac{3}{\sqrt{2}}e^{-3t}£^{-1}[\frac{\sqrt{2}}{(s+3)^2+(\sqrt{2})^2}]$$

$$= e^{-3t}\cos\sqrt{2}t - \frac{3}{\sqrt{2}}e^{-3t}\sin\sqrt{2}t$$

2. Second shifting (translation) theorem (shifting in t-variable)

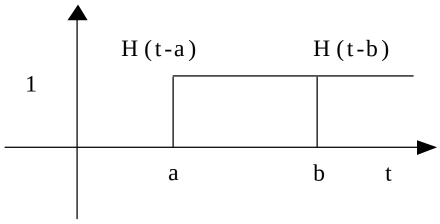
(1) Heaviside function H(t-a)

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



(2) Pulse function

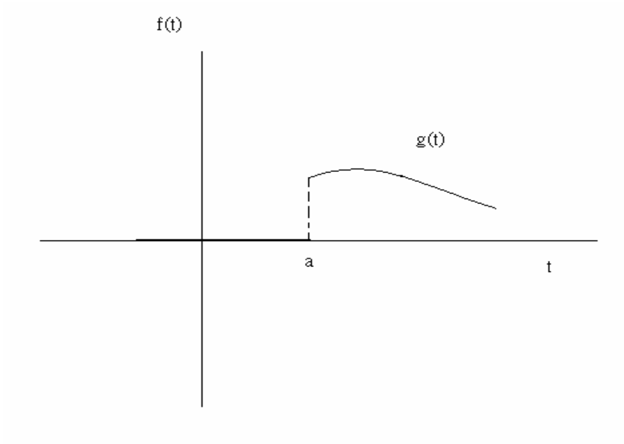
$$H(t-a)-H(t-b)=\begin{cases}0 & t<a\\ 1 & a\le t\le b\\ 0 & t>b\end{cases}$$




(3) Piecewise function in compact form

A: Piecewise function

$$f(t)=\begin{cases}0 & t<a\\ g(t) & t\ge a\end{cases}$$

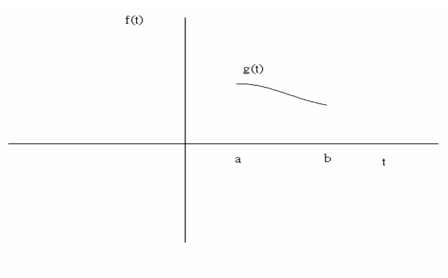



How to write f(t) in compact form?

f(t)=g(t) H(t-a)

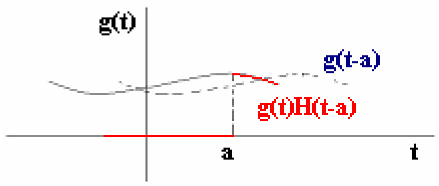
B. Piecewise function

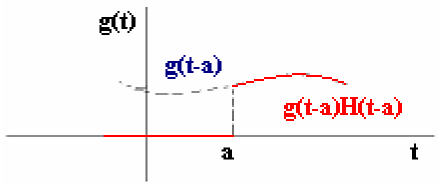
$$f(t)=\begin{cases}0 & t<a\\ g(t) & a\le t\le b\\ 0 & t>b\end{cases}$$




f(t)=g(t)[H(t-a)-H(t-b)]

(4) Shift function g(t-a)H(t-a)




(5) Second shift theorm

£$[f(t-a)H(t-a)] = e^{-as}F(s)$ for s>b

if £$[f(t)] = F(s)$ for s>b

proof:

£[f(t-a)H(t-a)]=$\int_0^\infty f(t-a)H(t-a)e^{-st}dt$

$$=\int_a^\infty f(t-a)e^{-st}dt$$

Let v=t-a, dv=dt , then

$\therefore$£[f(t-a)H(t-a)]=$\int_0^\infty e^{-(v+a)s} f(v)dv$

$$= e^{-as}\int_0^\infty e^{-sv} f(v)dv$$

$= e^{-as}F(s)$

Inverse form

$£^{-1}[e^{-as}F(s)] = f(t-a)H(t-a)$

and $£^{-1}[F(s)] = f(t)$

Ex: £[H(t-a)]=?

Sol: £[1 • H(t-a)]    f(t-a)=1

∴ According to 2nd shift theorm

£[1 • H(t-a)]= $e^{-as}$ £[1]= $\frac{1}{s}e^{-as}$

or £[H(t-a)]= $\int_0^\infty H(t-a)e^{-as}dt$

$= \int_0^\infty e^{-st}dt$

$= \frac{1}{s}e^{-as}$

Ex. solve: y''+4y=f(t)={ $\begin{matrix} 0 & t<3 \\ t & t\geq 3 \end{matrix}$

y(0)=y'(0)=0

Sol.

write f(t) in compact form

f(t)=tH(t-3)

∴ y''+4y=tH(t-3)

Take the Laplace transform

£[y'']+ £[y]= £[tH(t-3)]

$s^2\bar{y}(s) + 4\bar{y}(s) =$ £[tH(t-a)]

$=£[(t-3+3)H(t-3)]$

$=£[(t-3)H(t-3)]+3£[H(t-3)]$

$$=e^{-3s}\,£[t]+3\frac{e^{-3s}}{5}$$

$$=\frac{e^{-3s}}{s^2}+3\frac{e^{-3s}}{s}$$

$$\therefore \bar{y}(s)=\frac{1}{s^2(s^2+4)}e^{-3s}+\frac{3}{s(s^2+4)}e^{-3s}$$

$$=\frac{3}{4}\frac{1}{s}e^{-3s}-\frac{3}{4}\frac{s}{s^2+4}e^{-3s}+\frac{1}{4}\frac{1}{s^2}e^{-3s}-\frac{1}{4}\frac{1}{s^2+4}e^{-3s}$$

$y(t)= £^{-1}[\bar{y}(s)]$

$$=\frac{3}{4}£[^{-1}e^{-3s}\frac{1}{s}]-\frac{3}{4}£^{-1}[e^{-3s}\frac{s}{s^2+4}]+\frac{1}{4}£^{-1}[e^{-3s}\frac{1}{s^2}-\frac{1}{4}£$$

$$^{-1}[e^{-3s}\frac{1}{s^2+4}]$$

According to the inverse form of $2^{nd}$ shift theorm

$£^{-1}[e^{=as}F(s)]=f(t-a)H(t-a)$

$$y(t)=\frac{3}{4}H(t-3)\,£^{-1}[\frac{1}{s}]_{t\to t-3}-\frac{3}{4}H(t-3)\,£[\frac{5}{s^2+2^2}]_{t\to t-3}+$$

$$+\frac{1}{4}H(t-3)\,£^{-1}[\frac{1}{s^2}]_{t\to t-3}-\frac{1}{4}H(t-3)\frac{1}{2}£[\frac{2}{s^2+2^2}]_{t\to t-3}$$

$$=\frac{3}{4}H(t-3)1-\frac{3}{4}H(t-3)\cos 2(t-3)+$$

$$+\frac{1}{4}H(t-3)(t-3)-\frac{1}{8}H(t-3)\sin 2(t-3)$$

3.4 Convolution

1. Convolution

(f*g)(t)=$\int_0^t f(t-\tau)g(\tau)d\tau$ for t$\geq$ 0

When f(t) and g(t) are piecewise continuous on $[0,\infty)$

2. Convolution Theorem

L[f*g]=L[f(t)]L[g(t)]=F(s)G(s) if f*g is defined.

3. Inverse form

L$^{-1}$[F(s)G(s)]=f*g

where L$^{-1}$[F(s)]=f(t) & L$^{-1}$[G(s)]=g(t)

Ex. L$^{-1}$[$\frac{1}{s(s-4)^2}$]

Sol. $L^{-1}[\frac{1}{s}\frac{1}{(s-4)^2}]=L^{-1}[F(s)G(s)]$

F(s)=$\frac{1}{s}$ , L$^{-1}[F(s)]=f(t)=1$

G(s)=$\frac{1}{(s-4)^2}$, L[G(s)]=G(t)= $e^{4t}L^{-1}[\frac{1}{s^2}]=te^{4t}$

$$\therefore L^{-1}[\frac{1}{s(s-4)^2}]=f*g=\int_0^t f(t-\tau)g(\tau)d\tau=\int_0^t 1\tau e^{4\tau}d\tau$$

$$=\int_0^t \frac{1}{4}\tau d\tau(e^{4\tau})=\frac{1}{4}te^{4t}\Big|_0^t-\frac{1}{4}\int_0^t e^{4\tau}d\tau$$

$$=\frac{1}{4}te^{4t}-\frac{1}{16}e^{4\tau}\Big|_0^t=\frac{1}{4}te^{4t}-\frac{1}{16}e^{4\tau}+\frac{1}{16}$$

4. Commutative

$$f*g=g*f$$
$$\int_0^t f(t-\tau)g(\tau)d\tau=\int_0^t g(t-\tau)f(\tau)d\tau$$

Note: Integral defining the convolution of f and g may be easier to evaluate in one order than in the other.

5. Application

(1) Solve the solution of general initial-value problem

$$y''+Ay'+By = f(t); y(0) = C, y'(0) = D$$

(2) Solve integral equation with function to be determined in the integral

$$f(t) = q(t) + \int_0^t f(t-\tau)g(\tau)d\tau \quad \text{(f(t) is unknown)}$$

Ex.

$$f(t) = 2t^2 + \int_0^t f(t-\tau)e^{-\tau}dt$$
$$f(t) = ?$$

Sol.

$$L[f(t)] = L\left[2t^2\right] + L\left[f(t-\tau) * e^{-\tau}\right]$$
$$F(s) = \frac{4}{s^3} + L[f(t)]L\left[e^{-\tau}\right] = \frac{4}{s^3} + \frac{F(s)}{s+1}$$
$$\frac{s}{s+1}F(s) = \frac{4}{s^3}$$
$$F(s) = \frac{4(s+1)}{s^4} = \frac{4}{s^3} + \frac{4}{s^4}$$
$$f(t) = L[F(s)] = 2t^2 + \frac{2}{3}t^3$$

6. Other properties

(1) Distributive law

$$f * (g+h) = f * g + f * h$$

(2) Associative law

$$f * (g * h) = (f * g) * h$$

(3) $0 * f = f * 0 = 0$

3.5 Unit Impulses and Dirac Delta Function

1. Unit Impulse

$\delta_\varepsilon(t) = {}^1/\varepsilon\, [H(t)-H(t-\varepsilon)]$

and $\int_{-\infty}^{\infty} \delta_\varepsilon(t)dt=1$

$\delta_\varepsilon(t-a) = {}^1/\varepsilon\, [H(t-a)-H(t-a-\varepsilon)]$

$\int_{-\infty}^{\infty} \delta_\varepsilon(t-a)dt =1$ , unit impulse

2. Dirac Delta Function

$\delta(t) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t)$

$\delta(t-a) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-a)$

Properties:

(1) $\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$

(2) $\int_{0}^{\infty} \delta(t-a)dt = 1$

3. Laplace Transfer of Delta Function $\delta(t-a)$

$L[\delta(t-a)] = L[\lim_{\varepsilon\to 0} \delta_\varepsilon(t-a)] = \lim_{\varepsilon\to 0} L[\delta_\varepsilon(t-a)]$

Recalling unit step function

$\delta_\varepsilon(t-a) = {}^1/\varepsilon\, [H(t-a)-H(t-a-\varepsilon)]$

From 2nd translation theorem , we have

$L[\delta_\varepsilon(t-a)]$

$= {}^1/\varepsilon\; L[H(t-a)] - {}^1/\varepsilon\, [H(t-a-\varepsilon)]$

$= {}^1/\varepsilon\; e^{-as} L[1] - {}^1/\varepsilon\; e^{-(a+\varepsilon)s} L[1]$

$= {}^1/\varepsilon \cdot {}^1/_s\, [e^{-as} - e^{-(a+\varepsilon)s}]$

$= [e^{-as}(1-e^{-\varepsilon s})] / \varepsilon s$

$\therefore \lim_{\varepsilon \to 0} L[\delta_{\varepsilon}(t-a)] = \lim_{\varepsilon \to 0} [e^{-as}(1-e^{-\varepsilon s})] / \varepsilon s$

$= \lim_{\varepsilon \to 0} [(e^{-as} \cdot se^{-\varepsilon s}) / s]$

$= e^{-as}$

Thus $L[\delta(t-a)] = e^{-as}$

If a = 0 $L[\delta(t-a)] = 1$

4. Filtering Property

$\int_0^{\infty} f(t)\delta(t-a)dt = f(a)$

if f(t) is continuous function on $[0, \infty)$

EX. If $f(t) = e^{-st}$

$\therefore \int_0^{\infty} e^{-st}\delta(t-a)dt = e^{-sa}$ ( t = a)

根據 L T 定義 $L[\delta(t-a)] = e^{-sa}$

EX $\int_0^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$

Convolution formula $f * \delta = f(t)$

$\therefore \delta(t)$ acts as an identity for the convolution operation.

5. I.V.P involving $\delta$ function

$y''+2y'+2y = \delta(t-\varepsilon)$ $y=(0) = y'(0) = 0$

sol: Take Laplace transfer to obtain

$L[y'']+2L[y']+2L[y] = L[\delta(t-\varepsilon)] = e^{-3s}$

$(s^2+2s+2)\ \overline{Y}(s) = e^{-3s}$

$\overline{Y}(s) = e^{-3s} / (s^2+2s+2)$

$y(t) = L^{-1}[\overline{Y}(s)]$

$= L^{-1}[e^{-3s} / (s^2+2s+2)]$

$= L^{-1}[e^{-3s} / (s+1)^2+1]$

$= H(t-3)\ \ L^{-1}[1 / (s+1)^2+1]_{t \to t-3}$

$= H(t-3)[e^{-t} \sin t]_{t \to t-3}$

$= H(t-3)\, e^{-(t-3)} \sin (t-3)$

Hint:

$L^{-1}[1 / (s+1)^2+1] = e^{-t} \sin t$

3.6 Laplace transform solution of systems

1. system equations

$$\begin{cases} x'' - 2x' + 3y' + 2y = 4 \\ 2y' - x' + 3y = 0 \end{cases}$$

I.C. $x(0) = x'(0) = y(0) = 0$

$x(t) = ?\, y(t) = ?$

Take Laplace transform to yield

$$s^2 \bar{x}(s) - 2s\bar{x}(s) + 3s\bar{y}(s) + 2\bar{y}(s) = £[4] = \frac{4}{s}$$

$$2s\bar{y}(s) - s\bar{y}(s) + 3\bar{y}(s) = 0$$

$$\begin{cases} (s^2 - 2s)\bar{x}(s) + (3s+2)\bar{y}(s) = \dfrac{4}{s} & (1) \\ -s\bar{x}(s) + (2s+3)\bar{y}(s) = 0 & (2) \end{cases}$$

(1)+(2)*(s+2)

$$[(3s+2) + (s-2)(2s+3)]\bar{y}(s) = \frac{4}{s}$$

$$\bar{y}(s) = \frac{4}{s(2s^2 + 2s - 4)} = \frac{2}{s(s^2 + s - 2)} = \frac{2}{s(s+2)(s-1)} \quad \#$$

$$\bar{x}(s) = \frac{(2s+3)}{s}\bar{y}(s) = \frac{(4s+6)}{s^2(s+2)(s-1)} \quad \#$$

$$\therefore x(t) = £^{-1}[\bar{x}(s)] = £^{-1}\left[\frac{4s+6}{s^2(s+2)(s+1)}\right]$$

$$\frac{4s+6}{s^2(s+2)(s-1)} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S+2} + \frac{D}{S-1} = \frac{-7}{2}\frac{1}{S} - 3\frac{1}{S^2} + \frac{1}{6}\frac{1}{S+2} + \frac{10}{3}\frac{1}{S-1}$$

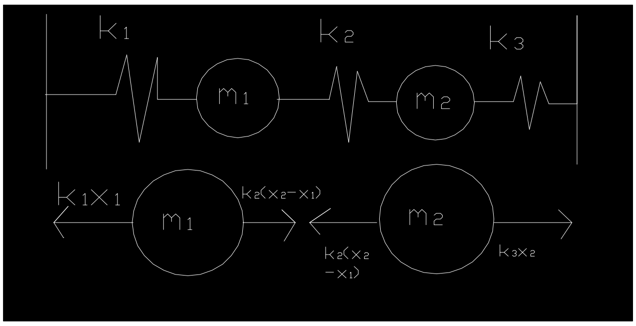
$$\therefore X(t) = \frac{-7}{2} - 3t + \frac{1}{6}e^{-2t} + \frac{10}{3}e^{t} \#$$

$$y(t) = £^{-1}\left[\bar{y}(s)\right] = £^{-1}\left[\frac{2}{s(s+2)(s-1)}\right]$$

$$\frac{2}{s(s+2)(s-1)} = \frac{A}{S} + \frac{B}{S+2} + \frac{C}{S-1} = \frac{-1}{S} + \frac{1}{S}\frac{1}{S+2} + \frac{2}{3}\frac{1}{S-1}$$

$$\therefore\ y(t) = £^{-1}\left[\frac{-1}{s}\right] + \frac{1}{3}£^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{3}£^{-1}\left[\frac{1}{s-1}\right] = -1 + \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t} \quad \#$$

2. Multiple mass – spring system




From Newton's 2nd law

$$m_1\ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1x_1$$
$$m_2\ddot{x}_2 = f_2(t) - k_2(x_2 - x_1) - k_3x_2$$

$\therefore$ system equations of motion are

$$\{m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t)$$
$$\{m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2k_1 = f_2(t)$$

initial conditions $\begin{cases} x_1(0) = s_1 & \dot{x}_1(0) = v_1 \\ x_2(0) = s_2 & \dot{x}_2(0) = v_2 \end{cases}$

3.7 Differential equations with polynomial coefficients

1. Derivatives of transform $\pounds\left[t^n f(t)\right] \qquad n \in N^+$

$$\frac{dF(s)}{ds} = ?$$

$$\frac{d^n F(s)}{ds^n} = ?$$

$$\pounds\left[f(t)\right] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

$$= -\int_0^\infty e^{-st}\left[tf(t)\right]dt$$

$$= -\pounds\left[tf(t)\right]$$

$$\therefore \pounds\left[tf(t)\right] = -\frac{dF(s)}{ds}$$

$$F(s) = \pounds\left[f(t)\right]$$

$$\frac{d^2F(s)}{ds^2} = -\frac{d}{ds}\int_0^\infty e^{-st} tf(t)dt$$

$$= \int_0^\infty e^{-st} t^2 f(t)dt$$

$$= \pounds\left[t^2 f(t)\right]$$

$$\pounds\left[t^2 f(t)\right] = \frac{d^2F(s)}{ds^2}$$

Similarly $\pounds\left[t^3 f(t)\right] = (-1)^3 \dfrac{d^3F(s)}{ds^3}$

$$\vdots$$

$$\vdots$$

$$\pounds\left[t^n f(t)\right] = (-1)^n \frac{d^nF(s)}{ds^n}$$

Ex： $£\left[t\sin at\right] = ?$

According to derivative of transform, we have

$$£\left[t\sin at\right] = -\frac{dL\left[\sin at\right]}{ds}$$

$$£\left[\sin at\right] = \frac{a}{s^2 + a^2}$$

$$\therefore £\left[t\sin at\right] = -\frac{d}{ds}\left[\frac{a}{s^2 + a^2}\right] = \frac{2as}{\left(s^2 + a^2\right)^2}$$

2. D.E with polynomial coefficient

D.E

$$ty'' + (4t - 2)y' - 4y = 0 \qquad y(0) = 1,\ y'(0)=0$$

sol：Take Laplace transform

$$£\left[ty''\right] + £\left[(4t-2)y'\right] - 4£\left[y\right] = 0$$

$$£\left[ty''\right] + 4£\left[ty'\right] - 2£\left[y'\right] - 4£\left[y\right] = 0 \qquad (1)$$

$$£\left[ty''\right] = -\frac{dL\left[y''\right]}{ds}$$

$$= -\frac{d}{ds}\left[s^2Y(s) - sy(0) - y'(0)\right]$$

$$= -\frac{d}{ds}\left[s^2Y(s) - s\right] \qquad (2)$$

$$= -s^2Y'(s) - 2sY(s) + 1$$

$$£\left[ty'\right] = -\frac{dL\left[y'\right]}{ds} = -\frac{d}{ds}\left[sY(s) - y(0)\right]$$

$$= -sY' - Y \qquad (3)$$

$$£\left[y'\right] = sY(s) - y(0) = sY - 1 \qquad (4)$$

$\therefore$ Substituting equations (2)～(4) into equation (1) yields

$$Y' + \frac{4s+8}{s(s+4)}Y = \frac{3}{s(s+4)} \qquad (5)$$

------------- linear  $1^{st}$ – order D.E  $(y'(x) + p(x)y) = q(x)$

Integrating factor $\mu(s)$

$$\mu(s) = e^{\int \frac{4s+8}{s(s+4)}ds} = e^{\ell n\left(s^2(s+4)^2\right)} = s^2(s+4)^2$$

Multiply equation (5) by $\mu(s)$ to obtain

$$\left(s^2(s+4)^2 Y\right)' = s^2(s+4)^2\frac{3}{s(s+4)} = 3s(s+4)$$

$$\therefore s^2(s+4)^2 Y(s) = \int 3s(s+4)ds + c$$

$$= s^3 + 6s^2 + c$$

$$Y(s) = \frac{s}{(s+4)^2} + \frac{6}{(s+4)^2} + c\left[\frac{a}{s} + \frac{b}{s^2} + \frac{e}{s+4} + \frac{f}{(s+4)^2}\right]$$

$a = -\frac{1}{3^2}$ , $b = \frac{1}{16}$ , $c = \frac{1}{3^2}$ , f= $\frac{3}{16}$

$y(t) =$ £$^{-1}$[Y(s)]