Ⅷ. Residuce Integration Mathod

1. Integrals of the from $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

   $F(\cos\theta, \sin\theta)$ = rational function of $\cos\theta$ & $\sin\theta$

   method

   Convert the real integral into complex integral by letting

   $z = \cos\vartheta + i\sin\vartheta = e^{i\vartheta} \quad 0 \le \vartheta \le 2\pi$

   $dz = ie^{i\vartheta}d\vartheta = izd\vartheta$

   $\therefore \; d\vartheta = \dfrac{dz}{iz}$

   And $\begin{cases} \cos\vartheta = \dfrac{e^{i\vartheta} + e^{-i\vartheta}}{2} = \dfrac{z+z^{-1}}{2} \\ \sin\vartheta = \dfrac{e^{i\vartheta} + e^{-i\vartheta}}{2} = \dfrac{z-z^{-1}}{2i} \end{cases}$

   $\therefore \; I = \int_0^{2\pi} F(\cos\vartheta, \sin\vartheta)d\vartheta = \oint_C F(\frac{1}{2}(z - z^{-1}))\frac{dz}{iz} \qquad C: |z| = 1$

EX Evaluate $\int_0^{2\pi} \dfrac{1}{(2+\cos\vartheta)^2} d\vartheta$

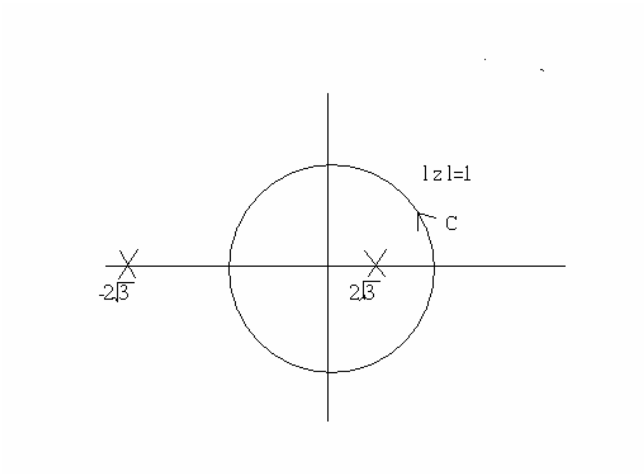
Sol~

The complex integral form of the above real integral is

$I = \dfrac{4}{\tau}\int_c \dfrac{z}{(z^2+4z+1)^2}dz \quad c = |z| = 1$

Poles of $\dfrac{z}{(z^2+4z+1)^2}$ are z =-2 $\pm\sqrt{3}$ and only -2+$\sqrt{3}$ is insider the unit circle C.

Hence,according Residuce Theorem, we have



$$\int_c \frac{z}{(z^2+4z+1)^2}dz = 2\pi i \operatorname*{Res}_{z\to -2+\sqrt{3}} f(z)$$

$\because$ -2+$\sqrt{3}$ is a pole of order 2.

$$\therefore \operatorname*{Res}_{z\to -2+\sqrt{3}} f(z) = \lim_{z\to 2+\sqrt{3}} \frac{d}{dz}\left[(z-(-2+\sqrt{3}))^2 \frac{2}{(z^2+4z+1)^2}\right]$$

$$= \lim_{z\to -2+\sqrt{3}} \frac{d}{dz}\left[\frac{z}{(z^2+2+\sqrt{3})^2}\right] = \frac{1}{6\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{1}{(2+\cos\vartheta)^2} d\vartheta = \frac{4}{i}\times\frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

2. Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

$f(x)$ is continuous on (∞,-∞)

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{R\to\infty}\int_0^R f(x)dx$$

(1) $\int_{-\infty}^{\infty} f(x)dx$ is convergent if both limits exist

(2) $\int_{-\infty}^{\infty} f(x)dx$ is divergent if one or both of the limits not exist

(3) If we know (a priori) that $\int_{-\infty}^{\infty} f(x)dx$ converges ,then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{Re\to\infty}\int_{-R}^{R} f(x)dx \ \ldots$$ (1) symmetric limit

NOTE: symmetric limit in Eq(1) may exist even though $\int_{-\infty}^{\infty} f(x)dx$ is divergent.

EX $\int_{-\infty}^{\infty} xdx$

Sol~

$$\int_{-\infty}^{\infty} xdx = \lim_{r\to\infty}\int_{-r}^{0} xdx + \lim_{R\to\infty}\int_0^R xdx \quad \Rightarrow \text{divergent}$$

But, symmetric limit

$$\lim_{R\to\infty}\int_{-R}^{0} xdx = \lim_{R\to\infty}\frac{x^2}{2}\Big|_{-R}^{R} = \left[\frac{R^2}{2} - \frac{(-R)^2}{2}\right] = 0$$

(4) Cauchy principal value

$$P.V\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

(a) if $\int_{-\infty}^{\infty} f(x)dx$ converges then $\int_{-\infty}^{\infty} f(x)dx = P.V\int_{-\infty}^{\infty} f(x)dx$

(b) if diverges $P.V\int_{-\infty}^{\infty} f(x)dx$ still exists

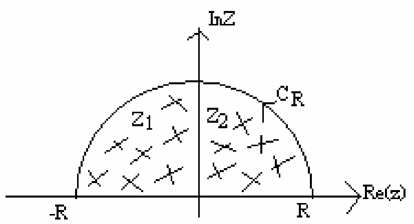
(5) Evaluation by Residue theory

$f(x) = \dfrac{P(x)}{Q(x)}$ continuous on $(-\infty,\infty)$

(a) Replace X by Z

$$f(z) = \frac{P(z)}{Q(z)}$$

(b) Integrate $f(z)$ over a closed contour C when C: $[-R,R]\cup C_R$ enclose all Poles of $f(z) = \dfrac{P(z)}{Q(z)}$ in upper-half plane $Re|z| > 0$



(c) Apply Residuce Theorem

$$\int_c f(z)dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z) = \int_{C_R} f(z) + \int_{-R}^{R} f(x)dx$$ Z_K :poles in upper-hlaf Plane

(d) Evaluate Cauchy principal vlue

$$P.V\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{Z=Z_K} f(z)$$

EX Evaluate $P.V\int_{-\infty}^{\infty}\dfrac{1}{(x^2+1)(x^2+9)}dx$

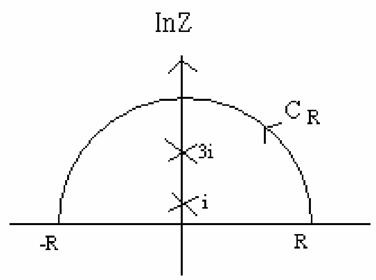
Sol~

$f(z) = \dfrac{1}{(z^2+1)(z^2+9)}$ pole $\pm i, \pm 3i$

$$\therefore \oint_c \frac{1}{(z^2+1)(z^2+9)}dz = 2\pi i(\underset{z=i}{Res}\ f(z) + \underset{z=3i}{Res}\ f(z)) = 2\pi i(\frac{1}{16i} - \frac{1}{48i}) = \frac{\pi}{12}$$

$$\oint_c f(z) = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}dz + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}dz \quad \text{as } R \to \infty$$

sufficient condition for $\int_{C_R} f(z)dz = 0$



The integral along C_R approaches zero as $R \to \infty$ when the derominator of $f(z)$ is of a power at least 2 more than its numerator

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}dx = \frac{\pi}{12}$$

(3) Integrals of the form $\int_{-\infty}^{\infty} f(x)dx\ cos\alpha xdx$ or $\int_{-\infty}^{\infty} f(x)dx\ sin\alpha xdx$

$f(x) = \frac{P(x)}{Q(x)}$ continuous on $(-\infty, \infty)$

Fourier Integral

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)cos\alpha xdx + i\int_{-\infty}^{\infty} f(x)sin\alpha xdx \quad \alpha > 0$$

Residuce methods

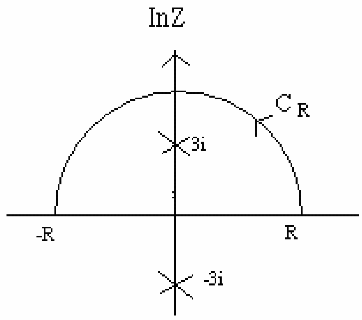
(a) Replace X by Z
 $f(z)e^{i\alpha z}dz$
(b) Integrate $f(z)e^{i\alpha z}$ over a closed contour C: $[-R, R] \cup C_R$ enclosing the poles of $f(z)$ in the upper plane

EX Evaluate the $P.V\int_0^{\infty} \frac{x\sin x}{x^2+9}dx$

Sol~

$$P.V\int_0^{\infty} \frac{x\sin x}{x^2+9}dx = \ {}^{1}\!/_{2}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}dx$$

$$\oint_c \frac{z}{z^2+9}e^{iz}dz = \int_{C_R}\frac{z}{z^2+9}e^{iz}dz + \int_{-R}^{R}\frac{x}{x^2+9}e^{ix}dx = 2\pi i \operatorname*{Res}_{Z\to 3i}\left(\frac{z}{z^2+9}e^{iz}\right) = \frac{\pi}{e^3}i$$



Pole of $\frac{z}{z^2+9}e^{iz}$ one $\pm 3i$

$\therefore$ $P.V\int_{-\infty}^{\infty}\frac{x}{x^2+9}e^{ix}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{x}{x^2+9}e^{ix} = \frac{\pi}{e^3}i$

$= P.V\int_{-\infty}^{\infty}\frac{x}{x^2+9}\cos xdx + \int_{-\infty}^{\infty}\frac{x}{x^2+9}\sin xdx$

$\therefore$ $P.V\int_{-\infty}^{\infty}\frac{x}{x^2+9}\cos xdx = 0$

$P.V\int_{-\infty}^{\infty}\frac{x}{x^2+9}\sin xdx = \frac{\pi}{e^3}$

(4) Indented contours

$\int_{-\infty}^{\infty} f(x)dx, \int_{-\infty}^{\infty} f(x)\cos\alpha xdx$ or $\int_{-\infty}^{\infty} f(x)\sin\alpha xdx$ in which $f(x) = \frac{P(x)}{Q(x)}$

have pales on the real axis

Supper f(x) has a simplex pole z=c on the real axis

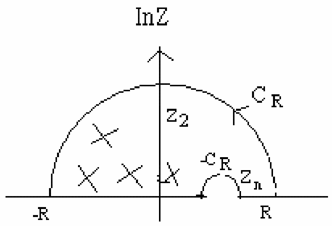
Evaluation of $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$

(a) Replace x by z

(b) Integrate $f(z)$ or $f(z)e^{i\alpha x}$ over a closed contour
C: $[-R, C-r]\cup[C+r, R]\cup C_R$ encloring the pole in upper-half plane

$$\oint_c f(z)dz = \int_{-R}^{C-r} f(z)dz + \int_{-C_r} f(z)dz + \int_{C+r}^{R} f(z)dz + \int_{C_R} f(z)dz$$

$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{Z=Z_k} f(z)$$



(c) $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$

$$\lim_{r\to\infty}\int_{C_R} f(z)dz = \pi i \operatorname*{Res}_{Z=C} f(z)$$

(4) Evaluate Cauchy principal value

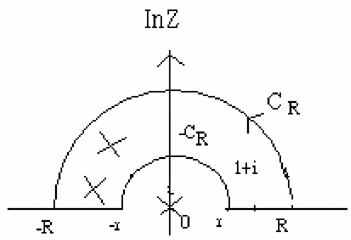
$$\text{P.V}\int_{-\infty}^{\infty} f(x)e^{ix}dx = \lim_{R\to\infty}\int_{-R}^{R} \text{f(x)}e^{i\alpha x}dx = 2\pi \text{i}\sum_{Z=Z_k}^{n} (f(z)e^{i\alpha z})$$

EX Evaluate $P.V\int_{-\infty}^{\infty}\frac{\sin\alpha}{x(x^2-2x+2)}dx$

Sol~

$$f(z) = \frac{1}{z(z^2-2z+2)} \quad \text{poles: } 0, \pm i$$

Evaluate $\oint_C \frac{1}{z(z^2-2z+2)}e^{iz}dz$ where C is chosen as follows



$$\therefore \oint_C = \oint_{C_R} + \int_{-R}^{r} + \int_{-C_r} + \int_{r}^{R}$$

$$= 2\pi i \operatorname*{Res}_{z=1+i}\left[\frac{e^{iz}}{z(z^2-2z+2)}\right] = 2\pi i(-\frac{e^{(-1+i)}}{4}(1+i))$$

As $R \to \infty$, $r \to 0$, we have

$$P.V\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx - \pi i \operatorname*{Res}_{z=0}\left[\frac{e^{iz}}{z(z^2-2z+2)}\right]$$

$$= 2\pi i\left[-\frac{e^{-1+i}}{4}(1+i)\right]$$

$$\therefore P.V\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx = \frac{\pi i}{2} + 2\pi i\left[-\frac{e^{-1+i}}{4}(1+i)\right]$$

$$\therefore P.V\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)}dx = \frac{\pi}{2}e^{-1}(\sin 1 + \cos 1)$$

$$P.V\int_{-\infty}^{\infty} \frac{\sin \alpha}{x(x^2-2x+2)}dx = \frac{\pi}{2}\left[1 + e^{-1}(\sin 1 - \cos 1)\right]$$