VII. Series and Residues

1. Sequence and series

(1) Sequences

(i) Definition

A Sequence { z_n }is a function whose domain is the set of positive integers.

EX: Sequence $\{1+i^n\}$ is 1+i, 0, 1-i, 2, 1+i, ………..

Somain n=1, n=2, n=3, n=4, n=5, ……

(ii) Convergence

A sequence { z_n } converges to a complex number L iff $R_e(z_n)$ and $I_m(z_n)$ converges to $I_m(L)$

$$\lim_{n\to\infty} z_n = \mathrm{L}$$

EX: $\{1+i^n\}$ divergent

EX: $\{\frac{i^{n+1}}{n}\}$ convergent sequence

$$\because \lim_{n\to\infty}\left(\frac{i^{n+1}}{n}\right) = 0$$

(2) Series

(i) Infinite Series

$$\sum_{k=1}^{\infty} Z_k = Z_1 + Z_2 + \ldots + Z_N + \ldots$$

(ii) Geometric Series

$$\sum_{k=11}^{\infty} aZ^{k-1} = a + aZ + aZ^2 + \ldots + aZ^{n-1} + \ldots$$

Special Geometric series

$$\frac{1}{1-Z} = 1 + Z + Z^2 + \ldots. \qquad |Z| < 1$$

$$\frac{1}{1+Z} = 1 - Z + Z^2 - Z^3 + \ldots \qquad |Z| < 1$$

(iii) convergence of series

If the sequence of partial sums {Sn} converges to L, the the series $\sum_{k=1}^{\infty} Z_k$ converges to L.

EX: Geometric series $\sum_{k=1}^{\infty} aZ^{k-1}$

Sol: n^{th} partial sum Sn of geometric series is Sn = a+Az+…..+aZ $^{n-1}$ = $\frac{a(1-Z^n)}{1-Z}$

{Sn}: Sequence of partial sums of series

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-Z^n)}{1-Z} = \frac{a}{1-Z} \quad \text{when} \quad |Z|<1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-Z^n)}{1-Z} \quad \text{diverges when} \quad |Z|\geq 1$$

(iv) convergence Test

A. Absolute convergence

The series $\sum_{k=1}^{\infty} Z_k$ converge absolutely if $\sum_{k=1}^{\infty} |Z_k|$ converges

B. Ratio Test

Suppose $\sum_{k=1}^{\infty} Z_k$ is a series of nonzero complex terms such that $\lim_{n\to\infty}\left|\frac{Z_{n+1}}{Z_n}\right| = L$

(a) If L<1, then the series converges absolutely

(b) If L>1, or L= ∞, then the series diverge

(c) If L=1, the test is incondusive

C. Root Test

Suppose $\sum_{k=1}^{\infty} Z_k$ is a series of complex terms such that $\lim_{n\to\infty}\sqrt[n]{|Z_n|} = L$

(a) same as B(a)

(b) same as B(b)

(c) same as B(c)

(3)Power Series

(i) Definition

A power series in $(Z - Z_0)$ is an infinite series of the form :

$$\sum_{k=0}^{\infty} q_k (Z - Z_0)^k = a_0 + a_1(Z - Z_0) + a_2(Z - Z_0) + ...a_k(Z - Z_0)^k \quad \text{where}$$

a_k = complex coefficient ; Z = center of the series

(ii)Circle and radius of convergence

If the series $\sum_{k=0}^{\infty} a_k (Z - Z_0)^k$ converges for $|Z - Z_0| < R$ and diverges for

$|Z - Z_0| > R$, where $0 < R < \infty$ the circle $|Z - Z_0| = R$ is called the circle of convergence, R is called the radius of convergence.

From ratio test the condition for convergence of power series

$\sum_{k=0}^{\infty} a_k (Z - Z_0)^k$ is

$$\lim_{k\to\infty}\left|\frac{q_{k+1}(z-z_0)^{k+1}}{q_k(z-z_0)^k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right||z-z_0| = L < 1$$

$$\therefore |z-z_0| < \frac{1}{\lim_{k\to\infty}\left|\frac{q_{k+1}}{q_k}\right|} = \frac{1}{L} = R$$

The radius of convergence R can be :

Radius of convergence

$R = 0 \quad (\mathrm{L} = \infty)$

$\mathrm{R} = 1/\mathrm{L} \quad (0 < \mathrm{L} < \infty)$

$\mathrm{R} = \infty \quad (\mathrm{L} = 0)$

※A similar remarks for root test by using $\lim_{n\to\infty}\sqrt[n]{|a_n|}$

Ex : power series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(z-1-i)^k}{k!}$ radius of convergence R = ?
circle of convergence ?

Sol : Using ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}/(n+1)!}{(-1)^{n+1}/n!}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 = L$$

$\therefore$ radius of convergence $\mathrm{R} = \frac{1}{\mathrm{L}} = \infty$

circle of convergence $|z - 1 - i| = \infty$

$\therefore$ series convergence for all z

Ex : power series $\sum_{k=1}^{\infty}(\frac{6k+1}{2k+s})(z-2i)^k$ $\mathrm{R} = ?, |z - z_0| = ?$

Sol : $a_n = (\frac{6k+1}{2k+s})^n$

root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{(\frac{6\text{n}+1}{2\text{n}+5})^n}=\lim_{n\to\infty}\frac{6\text{n}+1}{2\text{n}+5}=3=L$$

$\therefore$ Radius of convergence $\text{R}=\frac{1}{L}=\frac{1}{3}$

Circle of convergence $|z-2i|=\frac{1}{3}$

$\therefore$ Series convergence for $|z-2i|<\frac{1}{3}$

(iii) Properties of a power series within the circle of convergence

A. Continuity

A power series $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ represents a continuence function f(z)

B. Term-by-Term

A power series $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ can be differentiated term by term

C. Term-by-Term Integration

$\sum_{k=0}^{\infty}a_k(z-z_0)^k$ can be integrated term by term for any contour C in $|z-z_0|<R$

2. Taylor Series

(1) Taylor series

If f(z) is analytic in the interior of a circle with center Z_0 and radius R then $\forall|z-z_0|<R$ we have

$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k = f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{K!}(z-z_0)^k$ represented by power series ,which are called Taylor series of f(z) with center Z_0

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{K!}Z^k$$

(2) Taylor Theorem

Let f(z) be analytic within the domain D containing Z_0. The f(z) has the representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{K!}(z - z_0)^k$$

valid for the largest circle C with center of Z_0 and radius R contained in D

[proof]

z = fixed point with C

S= variable of integration

From Cauchy Integral theorem , we have

$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(s)}{s-z}ds$$

$$= \frac{1}{2\pi i}\oint_c \frac{f(s)}{(s-z_0)-(z-z_0)}ds = \frac{1}{2\pi i}\oint \frac{f(s)}{s-z_0}\left(\frac{1}{1-(\frac{z-z_0}{s-z_0})}\right)ds$$

$$= \frac{1}{2\pi i}\oint_c \frac{f(s)}{s-z_0}\left[1 + \frac{z-z_0}{s-z_0} + \left(\frac{z-z_0}{s-z}\right)^2 + \ldots\ldots + \left(\frac{z-z_0}{s-z_0}\right)^{n-1} + \frac{(s-z_0)}{(s-z)(s-z_0)^{n-1}}\right]ds$$

$$= \frac{1}{2\pi i}\oint_c \frac{f(s)}{s-z_0}ds + \frac{(z-z_0)}{2\pi i}\oint \frac{f(s)}{(s-z_0)^2}ds. + \frac{(z-z_0)^{n-1}}{2\pi i}\oint_c \frac{f(s)}{(s-z_0)^n}ds +$$

$$\frac{(z-z_0)^n}{2\pi i}\oint_c \frac{f(s)}{(s-z)(s-z_0)^n}ds$$

From Cauchy Integral formula for derivative We have

$$f''(z_0) = \frac{n!}{2\pi i}\oint \frac{f(s)}{(s-z_0)^{n+1}}ds$$

From Cauchy Integral formula for derivative

We have

$$\therefore f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f'''(z_0)}{1!}(z-z_0)^2 + \ldots.$$

$$+ \frac{f^{(n-1)}(z)}{(n-1)!}(z-z_0)^{n-1} + R_n(z)$$

$$R_n(z) = \frac{(z-z_0)^n}{(2\pi\, i)} \oint \frac{f(s)}{(s-z)(s-z_0)^n} ds$$

Now, show that $R_n(z) = 0$

From ML-inequality, we obtain

$$\left|R_n(z)\right| = \left|\frac{(z-z_0)}{2\pi\, i} \oint_c \frac{f(s)}{(s-z)(s-z_0)^n} ds\right| \le \left|\frac{(z-z_0)^n}{2\pi\, i}\right| \left|\frac{f(s)}{(s-z)(s-z_0)^n}\right| \left|\oint_c ds\right|$$

$\left|f(z)\right| < M,\ \left|s - z_0\right| = R,\ \left|z - z_0\right| = d\ ,\left|s - z\right| \ge R - d$

$$\therefore \left|R_n(z)\right| \le \frac{d^n}{2\pi} \frac{M}{(R-d)R^n} 2\pi R = \frac{MR}{R-d} (\frac{d}{R})^n$$

$\because (\frac{d}{R}) < 1\ \ (\frac{d}{R})^n \to 0\ \ \text{as } n \to \infty$

$\therefore \lim\limits_{n\to\infty} R_n(z) = 0$

$$\therefore\ f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + ...$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)} z_0}{k!} (z-z_0)^k$$

Ex : Expand $f(z) = \frac{1}{1-z}$ in Taylor series with center $z_0 = 2i$

Sol :

$$f(z) = \frac{1}{1-z}$$

$$f'(z) = \frac{1}{(1-z)^2}$$

$$f''(z) = \frac{1}{(1-z)^3}$$

$$f^n(z) = \frac{n!}{(1-z)^{n+1}}$$

Radius of convergence of above series ?

By ratio test , we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(1-2i)^{n+2}}}{\frac{1}{(1-2i)^{n+1}}}\right| = \lim_{n\to\infty}\frac{1}{|1-2i|} = \frac{1}{\sqrt{5}}$$

$\therefore$ Radius of converges R is $R = \dfrac{1}{1/\sqrt{5}} = \sqrt{5}$

Circle of converges is $|z - 2i| = \sqrt{5}$

$\therefore$ Series converges for $|z - 2i| < \sqrt{5}$

Ex : Expand $f(z) = \dfrac{1}{1-z}$ in a Taylar series with center $z_0 = 0$

Sol : From geometry series , we have

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots\ldots + z^n + ..$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{z_n}\right| = |z| < 1 \quad \text{for convergence}$$

3. Laurent series

(1) Annulus

A ring-shaped region between tow circles

(2) Open annulus A

An open region between circles $|z - z_0| = r$

And $|z - z_0| = R$ with r < R

$$A = \{z \mid r < |z - z_0| < R\}$$

Some possible annulus domain

r	R	domain	
0	finite	Interior of $|z - z_0| = R$ except Z_0	

$\neq 0$	∞	All points exterior to $\lvert z-z_0\rvert = r$	
0	∞	Entire complex plane except Z_0	
finite	finite	Points exterior to $\lvert z-z_0\rvert = r$ and interior to $\lvert z-z_0\rvert = R$	

(3) Laurent Series

Support that $\sum_{R=\infty}^{-1} q_k (z-z_0)^k$ converges in the region $\lvert z - z_0\rvert > r$ and $\sum_{R=\infty}^{-1} a_k (z-z_0)^k$ converges in the disk $\lvert z - z_0\rvert > R$. Both series converges in the open annulus $r < \lvert z-z_0\rvert < R$ The sum is written as

$$\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

which represent an analytic function in the annulus. All expansions of this type are called Laurent series

(4) Laurent theorem

Let f(z) be analytic within the annulus domain A defined by $1 < \lvert z-z_0\rvert < R$.Then f(z) has the series representation

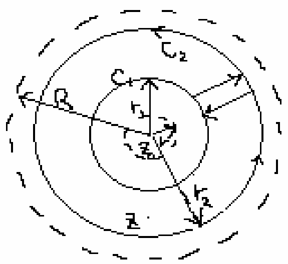
$$\mathrm{f(z)} = \sum_{k=-\infty}^{\infty} q_k (z-z_0)^k$$

valid for $r < \lvert z-z_0\rvert < R$ .coefficient a_k are given by

$$a_x = \frac{1}{2\pi\, i}\oint \frac{f(s)}{(s-z_0)^{k+1}}\,\mathrm{ds} \quad \mathrm{k} = \pm 0, \pm 1, \pm 2, \ldots\ldots.$$

When C is simple closed curve within A with z_0 in the interior

[proof]



$C_1 \,\&\, C_2 =$ circles with $\mathrm{r} < \mathrm{r}_1 < r_2 < R$ and $\mathrm{r}_1 < |z - z_0| < \mathrm{r}_2$

From Cauchy Integral formula

$$f(z) = \frac{1}{2\pi\, i}\oint_{c_2} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi\, i}\oint_{c_1} \frac{f(s)}{s-z}\,ds$$

$$\frac{1}{2\pi\, i}\oint_{c_2} \frac{f(s)}{s-z}\,ds = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

Where

$$a_k = \frac{1}{2\pi\, i}\oint_{c_1} \frac{f(s)}{(s-z_0)^{k+1}}\,ds \qquad \mathrm{k} = 0,1,2,..$$

$$-\frac{1}{2\pi\, i}\oint_{c_1} \frac{f(s)}{(s-z_0)^{k+1}}\,ds. = \frac{1}{2\pi\, i}\oint_{c_1} \frac{f(s)}{z-z_0})(\frac{1}{1-{s-z_0}\big/{z-z_0}}\,ds$$

$$= \frac{1}{2\pi\, i}\oint_{c_1} \frac{f(s)}{z-z_0})\{1 + \frac{s-z_0}{z-z_0} + (\frac{s-z_0}{z-z_0})^2 + \dots\dots + (\frac{s-z_0}{z-z_0})^{n-1} + \frac{(s-z_0)^n}{(z-s)(z-z_0)^{n-1}}\}ds$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k} + R_n(z)$$

Where

$$a_k = \frac{1}{2\pi\, i}\oint_{c_1} \frac{f(s)}{(s-z_0)^{k+1}}\,ds \qquad \mathrm{k} = 1,2,3$$

and

$$R_n(z) = \frac{1}{2\pi\, i(z-z_0)}\oint_{c_1} \frac{f(s)(s-z_0)^n}{z-s}\,ds$$

Now show that $\lim_{n\to\infty} |R_n(z)| \to 0$

Let $|z - z_0|$ = d , max($|f(z)|$ = M on C_1

Because $|s - z_0|$ = r_1

$$|z - s| = |z - z_0 - (s - z_0)| \geq |z - z_0| - |s - z_0| = d - r_1$$

ML-equality gives

$$|Rn(z)| = \left| \frac{1}{2\pi i (z - z_0)^n} \oint_c \frac{f(s)(s - z_0)^n}{z - s} ds \right| \leq \frac{1}{2\pi d^n} \frac{M r_1^n}{d - r_1} 2\pi r_1 \frac{M r_1}{d - r_1} (\frac{r_1}{d})^n$$

Because $r_1 \leq d$ $(\frac{r_1}{d})^n$ -> 0 as n -> ∞

So lim$|Rn(z)|$ -> 0

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{(s - z)} ds = \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k} \quad \sim\sim\sim(6)$$

Combining equations (2) and (6) , equation (1) yields

$$F(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k}$$

Or $\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$

And equations (3) and (5) can be written as a single integral

$$a_k = \frac{1}{2\pi i} \oint_c \frac{f(s)}{(z - z)^{k+1}} dz \qquad k = 0 .+1,-1,+2,-2.....$$

Remarks:

A. If a_k = 0 for k = -1, -2 …..

Laurent series is Taylor series . Laurent expansion a generalization of a Taylor series

B. The formula for the coefficients a_k of a Laurent series is seldom used

(a) geometric series : $\frac{1}{1-z}$, $\frac{1}{1+z}$ …..

(b) known series : Sin z , Cosz e^{z} ……

Expand f(z) = $\dfrac{1}{z-(z-1)}$ in a Laurent series valid for

(a) $0<|z|<1$ (b) $1<|z|$ (c) $0<|z-1|<1$ (d) $1<|z-1|<\infty$

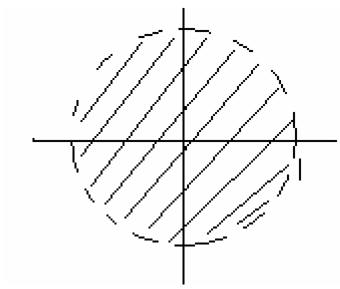
(a) f(z) = $-\dfrac{1}{z}(\dfrac{1}{1-z})$ = $-\dfrac{1}{z}(1+z+z^{2}+...)$ = $-\dfrac{1}{z}$ -(1+ $z+z^{2}+....)$

$-\dfrac{1}{z}$ converges for $|z|>0$

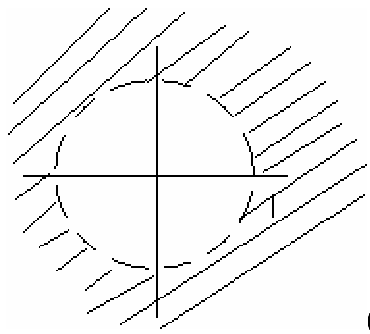
$1+z+z^{2}+.....$ converges for $|z|<1$

$\lim\limits_{m->\infty}\left|\dfrac{z^{n+1}}{z^{n}}\right|$ = $|z|<1$ for convergence

so $\dfrac{1}{z}$ -(1+ $z+z^{2}+....)$ converges for $0<|z|<1$



(a)



(b)

(b) f(z) = $\dfrac{1}{z}(\dfrac{1}{(z-1)}$ = $z^{2}(\dfrac{1}{1-\dfrac{1}{z}})$ = $\dfrac{1}{z^{2}}(1+\dfrac{1}{z}+\dfrac{1}{z^{2}}+....)$ = $\dfrac{1}{z^{2}}+\dfrac{1}{z^{3}}+....$

ratio test $\left|\dfrac{\dfrac{1}{z^{2+n+1}}}{z^{2+n}}\right|$ = $\left|\dfrac{1}{z}\right|<1$ for converges for series

so $1<|z|$