VI. Integral in the complex plane

1. Contour Integral

(1) Contour integral (complex line integral)

Let $f(z)=u(x,y)+iv(x,y)$ be defined along a piecewise smooth

curve (contour or path) c defined by $z=x+iy$

The contour integral of $f(z)$ along c is

$$\int_c f(z)dz=\int_c (u+iv)(dx+dy)=\int_c (udx-vdy)+i\int_c (vdx+udy)$$

$\rightarrow$Two real line integral

(2) Alternative form

Assume smooth curve c is defined by $z(t)=x(t)+iy(t)$ $,a\le t\le b$

Then

$$\int_c f(z)dz=\int_a^b [ux'(t)-vy'(t)]dt+i\int_a^b [vx'(t)+uy'(t)]dt$$

$$u=u(x(t),y(t)),v=(x(t)+y(t))$$

$$\therefore \int_c f(z)dz=\int_a^b [(u+iv)(x'(t)+iy'(t))]dt=\int_a^b f(z(t))z'(t)dt$$

$$(dz=\frac{dz}{dt}dt=z'(t)dt)$$

Ex: evaluate $\oint_c \frac{1}{z}dz$

$\mathrm{c:z(t)=cost+isint}$

Sol: method1:

$$\mathrm{z(t)=cost+isint}$$

$$\mathrm{z'(t)=-sin\,t+icost}$$

$$\therefore \oint_c \frac{dz}{z}=\int_0^{2\pi}\frac{z'(t)}{z(t)}dt=\int_0^{2\pi}\frac{-\sin t+i\cos t}{\cos t+i\sin t}dt=i\int_0^{2\pi}dt=2i\pi$$

method2:

$$\mathrm{z(t)=cost+isint=e^{it}}$$

$$z'(t)=ie^{it}$$

$$\therefore \oint_c \frac{dz}{z}=\int_0^{2\pi}\frac{z'(t)}{z(t)}dt=\int_0^{2\pi}\frac{ie^{it}}{e^{it}}dt=2i\pi$$

(3) Basic properties of complex line integral

A. Linear operation

$$\int_c [\alpha f(z) + \beta g(z)] dz = \alpha \int_c f(z) dz + \beta \int_c g(z) dz$$

B. Decomposition

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz \quad \text{if} \quad c = c_1 = c_2$$

C. Reversion

$$\int_{z_0}^{z_1} f(z)dz = -\int_{z_1}^{z_0} f(z)dz$$

(4) A bounding theorem (ML-inequality)

If $|f(z)| \le M$ for all z on c, then $\left|\int_c f(z)dz\right| \le ML$ where L is the length of c

Ex: $\int_c \frac{dz}{z}$ $\quad c : z(t) = \cos t + i \sin t \ , 0 \le t \le 2\pi$

Sol: $f(z) = \frac{1}{z}$

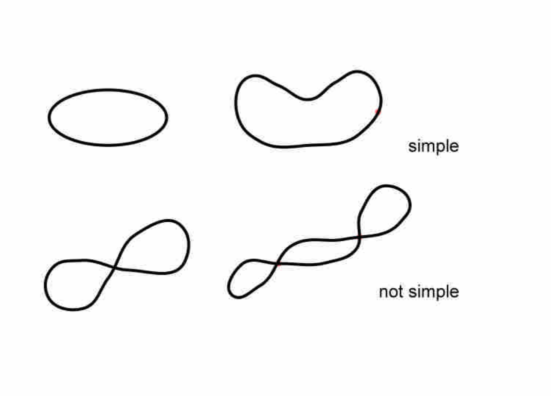
$|f(z)| = \left|\frac{1}{z}\right| = \frac{1}{|z|} \le 1 = M$ for all z on c

$\therefore$ According to ML-inequality, we have

$\therefore$ $\left|\int_c \frac{dz}{z}\right| \le ML = (1)(2\pi)$

2. Cauchy-Goursat Theorem (Cauchy Integral Theorem)

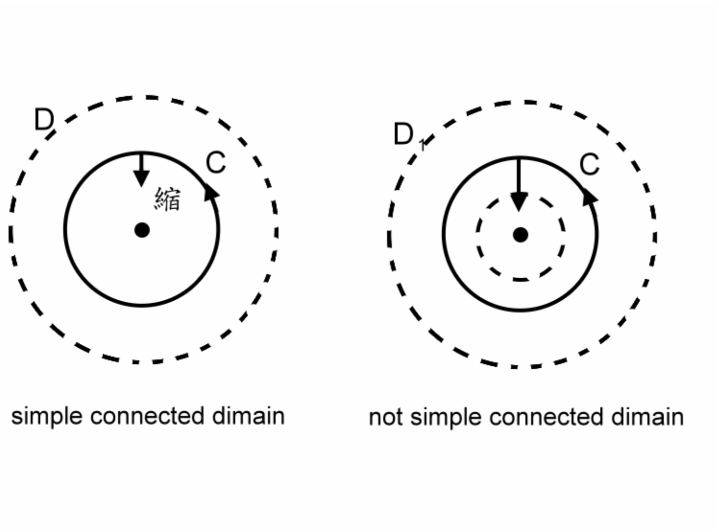
(1)Simple closed contour (path) a close path doesn't intersect itself



(2)Simple connected domain

Every simple closed contour C lying in domain D can be shrunk to a point

Without leasing D



(3)Cauchy-Goursat Theorem

A. Suppose a function $f(z)$ is analytic in a simple connected domain D,

Then for

Every simple closed path C in D

$$\oint_C f(z)dz = 0$$

B. If $f(z)$ in analytic at all points within and a simple closed contour

C, Then

$$\oint_C f(z)dz = 0$$

[Poof]

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

According to Green Theorem, we have

$$\int_C fdx + gdy = \int_D\int (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dA$$

$\therefore$

$$\int_C (udx - vdy) = \int_D\int (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dA$$

$$\int_C (vdx + udy) = \int_D\int (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dA$$

$\because$ $f(z)$ is analytic in D

$\therefore$Cauchy－Riemann equation are satisfied

Thus

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Hence

$$I_1 = 0$$
$$I_2 = 0$$

$\therefore$ $$\oint_C f(z)dz = 0$$

EX: Evaluate $\oint_C e^z dz = 0$ C: simple closed contour

(Sol)

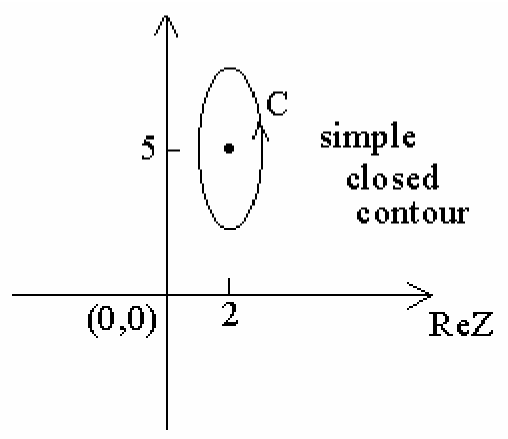
$\oint_C e^z dz = 0$ is entire function

e^z is analytic on entire complex

$\therefore$ Form Cauchy integral theorem, we have

$$\oint_C e^z dz = 0$$

Ex. Evaluate $\oint_c \frac{dz}{z^2}$ $c : (x-2)^2 + \frac{(y-5)^2}{4} = 1$



$f(z) = \frac{1}{z^2}$ analytic except z=0

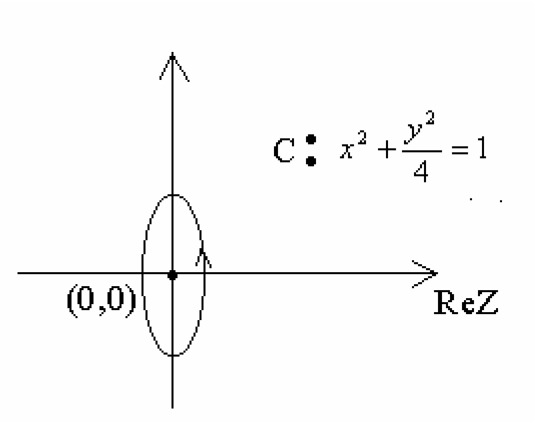
$\therefore f(z) = \frac{1}{z^2}$ is analytic within and on c.

Acconding to Cauchy integral-theorem, we have

$$\oint_c \frac{dz}{z^2} = 0$$

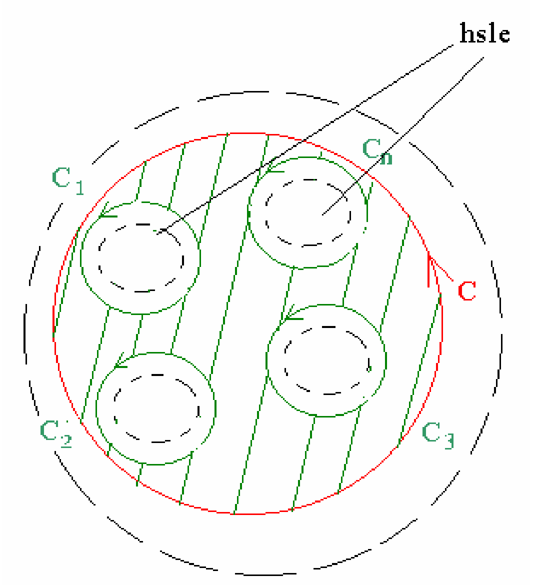
If the contour C is $x^2 + \frac{y^2}{4} = 1$,

Then $\oint_c \frac{dz}{z^2} = ?$



(4) multiple connected domain (MCD)

A domain is not simple connected called Multiple connected domain

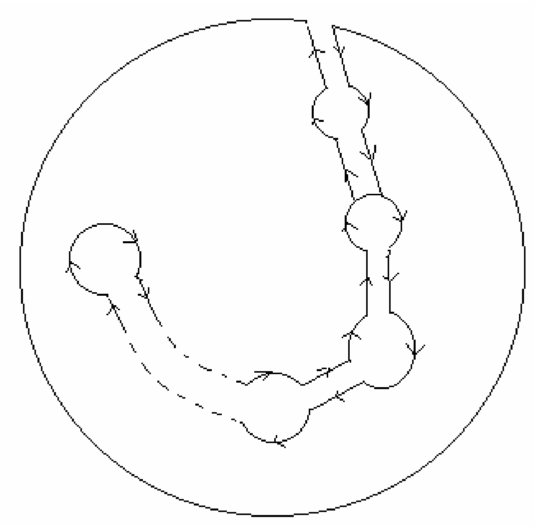


(5) Cauchy Integral Theorem for MCD Let $C_1, C_2, C_3 ... C_n$ be simple closed curves with positive orientation. Courves $C_1, C_2, C_3 ... C_n$ are interior to curve c and have no points in common.

If function $f(z)$ is analytic on each contour and at each point interior to C but exterior to all the C_n, then

$$\oint_c f(z)dz = \sum_{k=1}^{n} \oint_{c_k} f(z)dz$$

[proof]



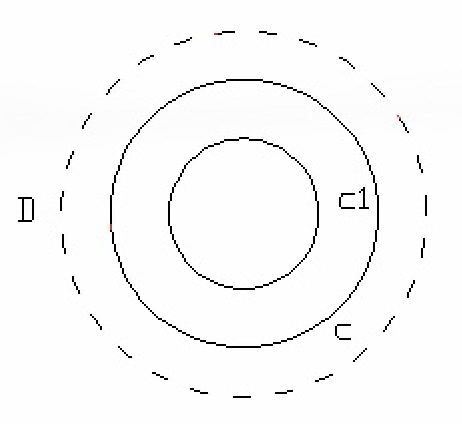
$$\therefore \int_{c-c_1-c_2--c_n} f(z)dz + \sum(\downarrow + \uparrow)$$

$$= \oint_c f(z)dz + \sum_{k=1}^{n} \oint_{C_k} f(z)dz + \int_{\sum(\downarrow+\uparrow)} f(z)dz \rightarrow 0$$

$= 0$ (Form Cauchy integnal theorem)

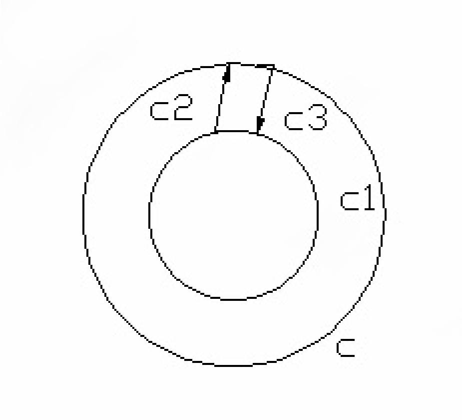
$$\oint_c f(z)dz = -\sum_{k=1}^{n} \oint_{c_k} f(z)dz = \sum_{k=1}^{n} \oint_{c_k} f(z)dz$$

Principle of Deformation of Contour



$$\oint_{c} f(z)dz = \oint_{c1} f(z)dz$$

[proof]



$$\because \int_{c1+c2+c3} f(z)dz$$
$$= \oint_{c1} f(z)dz + \int_{c2\uparrow} + \int_{c3\downarrow} + \oint_{c} f(z)dz$$
$$= 0$$

(Cachy integral theorem)

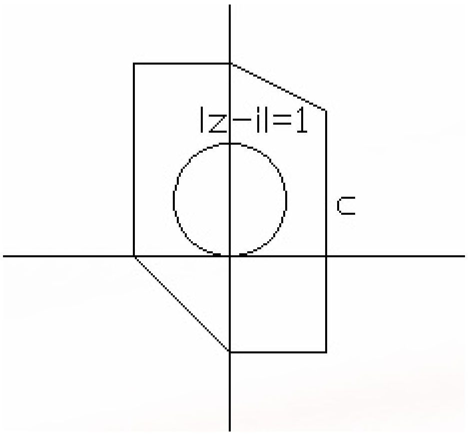
$$\therefore \oint_{c} f(z)dz = \oint_{c1} f(z)dz$$

[note]

This formula allow us to evaluate an integral over an complex simple closed contour by replacing a simple contour.

Ex：

Evaluate $I = \oint_{c} \frac{dz}{z-i}$ ， c is as shown



Sol

$f(z) = \frac{1}{z-i}$ is analytic except $z = i$

let contour $C_1 : |z-i| = 1$ enclosing

the point $z = i$

$\therefore f(z) = \frac{1}{z-i}$ is analytic between $C_1 \,\&\, C_2$ and on $C_1 \,\&\, C_2$

$\therefore$ from Cauchy-integral theorem for MCD，we have use complex line integral to evaluate $\oint_{C_1} \frac{dz}{z-i}$

$C_1 : |z = i| = 1$

$z - i = e^{it} \quad 0 \le t \le 2\pi$

$dz = i e^{it} dt$

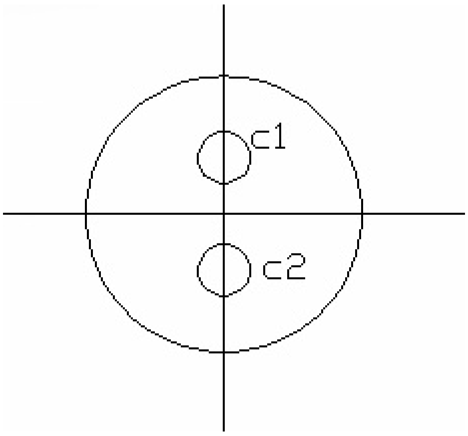
$$\therefore \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}\,dt}{e^{it}} = it\Big|_0^{2\pi} = 2\pi i$$

EX：

Evaluate $\oint_c \frac{dz}{z^2+1}$

$C : |z| = 3$

Sol



$f(z)=\frac{1}{z^2+1}$ is not analytic at $\pm i$

Take $\begin{cases} c_1 = |z-i| = \frac{1}{2} \\ c_2 = |z+i| = \frac{1}{2} \end{cases}$ with no common point

Form Cauchy integral theorem form MCD，we have

$$\oint_c \frac{dz}{z^2+1} = \oint_{C_1} \frac{dz}{z^2+1} + \oint_{C_2} \frac{dz}{z^2+1}$$

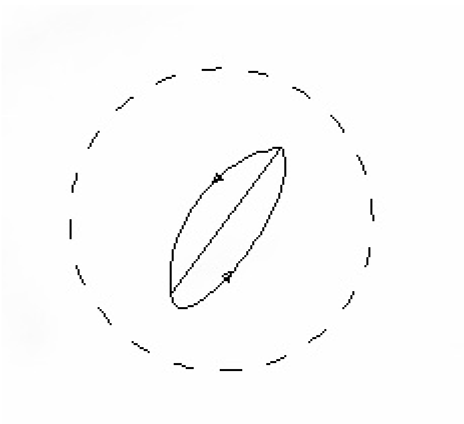
$$= \frac{1}{2i}\left[\oint_{C_1} \frac{1}{z-i} - \oint_{C_1} \frac{dz}{z+i}\right] + \frac{1}{2i}\left[\oint_{C_2} \frac{dz}{z-i} - \oint_{C_2} \frac{dz}{z+i}\right]$$

$$= \pi - \pi = 0$$

3.Independence of Path

(1) path independent

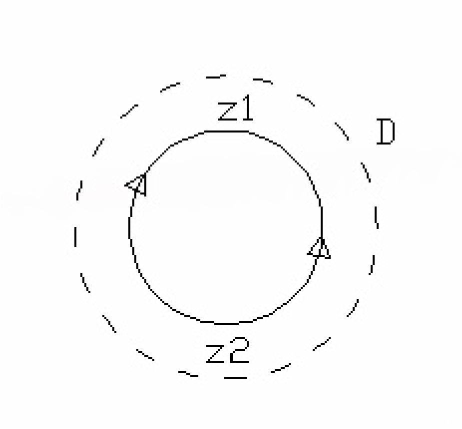
if $f(z)$ is an analytic function in a simple connected domain D，then $\int_c f(z)dz$ is independent of the path C in D



$$\int_c f(z)dz = F(z_1) - F(z)$$

where z_0 & z_1 are initial point and end point of contour C； $f(z)$ is an antiderivative (or indefinite integral) of $f(z)$ in D，$F'(z)=f(z)$

[proof]



contour c_1 & c_2 are continuous in a simple connected domain D，and c & c_1 from a closed contour

$\because f(z)$ is analytic in D，from Cauchy integral theorem，we have

$$\oint_{c-c_1} f(z)dz = 0$$

$$= \int_c f(z)dz + \int_{c_1} f(z)dz$$

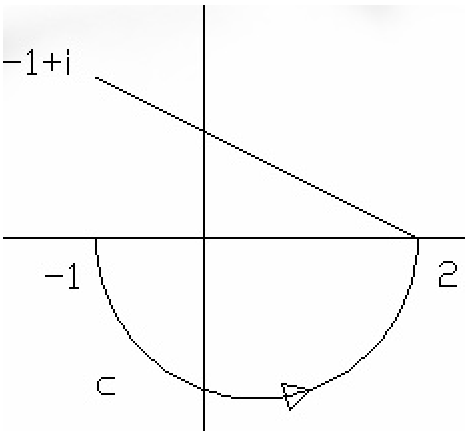
$$= \int_c f(z)dz - \int_{c_1} f(z)dz$$

$$\therefore \int_c f(z)dz = \int_{c_1} f(z)dz$$

independent of path

EX：

Evaluate $I = \int_c 2zdz$ C：given as shown



Sol

$f(z) = 2z$ is entire function

$\therefore I = \int_c 2zdz$ is independent of path

$$\therefore I = \int_c 2z dz = \int_{z_0=-1}^{z_1=-1+i} 2z dz = z^2 \Big|_{-1}^{-1+i}$$
$$= (-1+i)^2 - (-1)^2$$
$$= -1 - 2i$$

4.Cauchy Integral Formula

(1) consequence of Caucht-Goursat Theorem

A. The value of an analytic function $f(z)$ at z_0 in simple connected domain can represent by a contour integral

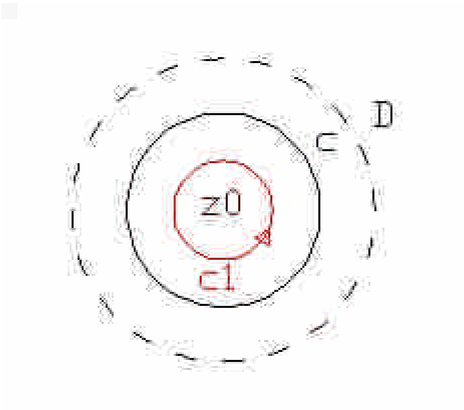
$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$$

B. An analytic function in a simply connected domain posses derivatives of all order

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$$

(2) Cauchy Integral Formula

If $f(z)$ is analytic at all point within and on a simple closed contour C lying within a simple connected domain D，and z_0 is any point interior to C，then $f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$



[proof]

By principle of deformation of contour

$$I = \oint_c \frac{f(z)}{z - z_0} dz$$
$$= \oint_{c_1} \frac{f(z)}{z - z_0} dz$$
$$= \oint_{c_1} \frac{f(z_0) + f(z) - f(z_0)}{z - z_0} dz$$
$$= f(z_0) \oint_{c_1} \frac{dz}{z - z_0} + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0} dz$$
$$= 2\pi i + I_1$$

By ML-integrality，and choose $c_1 : |z - z_0| = \frac{\delta}{2}$

$$\therefore |I_1| = \left| \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \left| \frac{f(z) - f(z_0)}{z - z_0} \right| 2\pi \left(\frac{\delta}{2} \right)$$
$$= \frac{|f(z) - f(z_0)|}{|z - z\ |} \pi\delta \leq \frac{\varepsilon}{\delta} \pi\delta = \pi\varepsilon$$

$\because f(z)$ is continuous at z_0，for small $\varepsilon > 0$，such that

$|f(z) - f(z_0)| < \varepsilon$ wherever $|z - z_0| < \delta$

$$\therefore \varepsilon \to 0 \Rightarrow I_1 = 0$$
$$\therefore \oint_c \frac{f(z)}{z - z_0} = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$$

(3) Cauchy Integral Formula for Derivative

If $f(z)$ is analytic in a simply connected domain D，and C is a simple closed contour lying within D，then

$$f(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{c(z - z_0)^{n+1}} dz$$

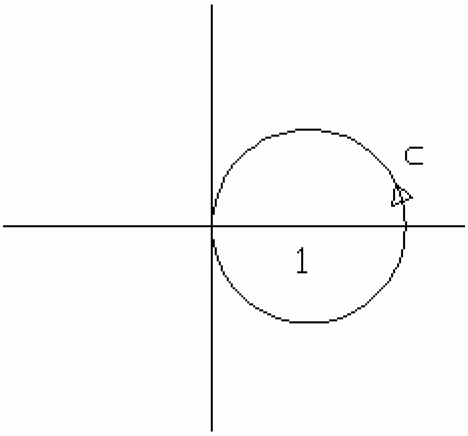
where z_0 is point interior to C

EX：

Evaluate $\oint_c \frac{z^2 + 1}{z^2 - 1} dz$ where C is unit circle with center at ① $z_0 = 1$ ② $z_0 = 1/2$ ③ $z_0 = -1$ ④ $z\ = i$

Sol

① $|z - 1| = 1$



$$\oint_c \frac{z^2+1}{z^2-1}dz = \oint_c \frac{1}{z-1}\left(\frac{z^2+1}{z+1}\right)dz$$

$\oint_c \frac{z^2+1}{z^2-1}dz$ is analytic within and on contour C:$|z-1|=1$，and $z_0=1$ is within C

$\therefore$ from Cauchy integral theorem，we have

$$\oint_c \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}dz = 2\pi i\left(\frac{z^2+1}{z+1}\right)_{z=1}$$

$$= 2\pi i\left(\frac{1^2+1}{1+1}\right)$$

$$= 2\pi i$$

$\oint_c \frac{z^2+1}{z^2-1}dz$ isn't analytic at $z=\pm 1$

C:$|z-1|=1$

$z-1=e^{ie}-0\le t\le 2\pi$

$$\oint_{c_1} f(z)dz = \frac{1}{2}\oint_{c_1}\left[\frac{z^2+1}{z-1}-\frac{z^2+1}{z+1}\right]dz$$

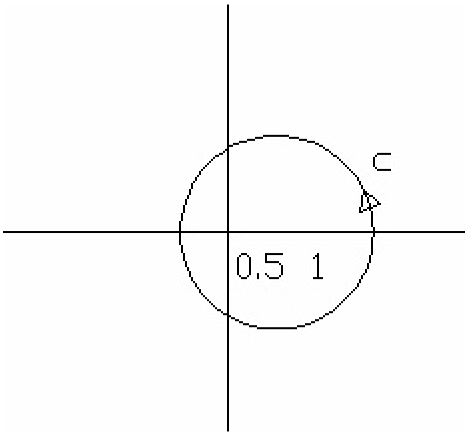
$$= \frac{1}{2}\oint_{c_1}\frac{z^2+1}{z-1}dz = \frac{1}{2}\int_0^{2\pi}\frac{(e^{it}+1)^2+1}{e^{it}}ie^{it}dt$$

$$= \frac{i}{2}\int_0^{2\pi}(e^{2it}+2e^{it}+2)dt$$

$$= \frac{i}{2}\left[\frac{1}{2i}e^{2it}+\frac{2}{i}e^{it}+2t\right]_0^{2\pi}$$

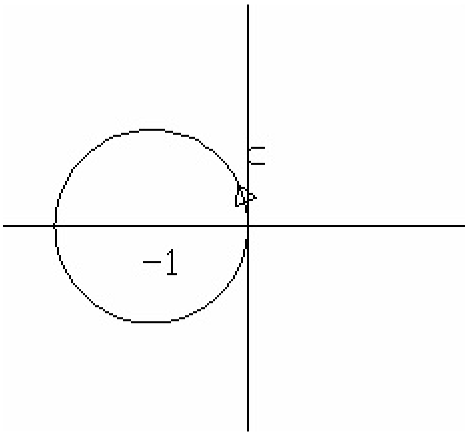
$$= 2\pi i$$

② $\left|z-\frac{1}{2}\right|=1$ containing $z=1$



$$\therefore \oint_c \frac{z^2+1}{z^2-1}dz = \oint_c \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}dz$$
$$= 2\pi i\left(\frac{z^2+1}{z+1}\right)_{z=1} = 2\pi i$$

③ $|z+1|=1$

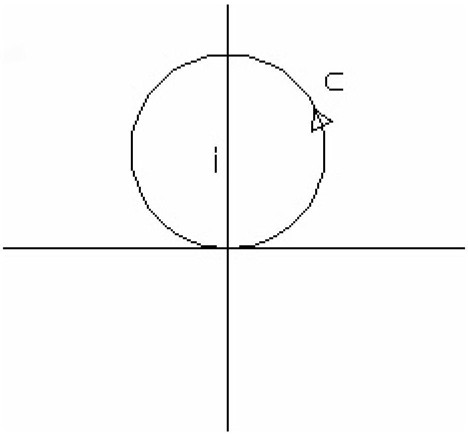


$$\oint_c \frac{z^2+1}{z^2-1}dz = \oint_c \frac{\left(\frac{z+1}{z-1}\right)}{z+1}dz$$

$f(z)=\frac{z+1}{z-1}$ is analytic within and on contour $|z+1|=1$

$$\therefore \oint_c \frac{z^2+1}{z^2-1}dz = 2\pi i\left(\frac{z^2+1}{z-1}\right)$$
$$= 2\pi i\left(\frac{(-1)^2+1}{-1-1}\right)$$
$$= 2\pi i$$

④ $|z-i|=1$



$\because \dfrac{z^2+1}{z^2-1}$ is analytic within and on contour $|z-i|=1$

$\therefore$ from Cauchy integral theorem

$$\oint_c \frac{z^2+1}{z^2-1}dz=0$$