V. Complex Number and Function

1. Complex Number

（1） complex number

$Z = x + i\, y \qquad i^2 = -1$

Re（z）= x …… Real part

Im（z）= y …… Imaginary part

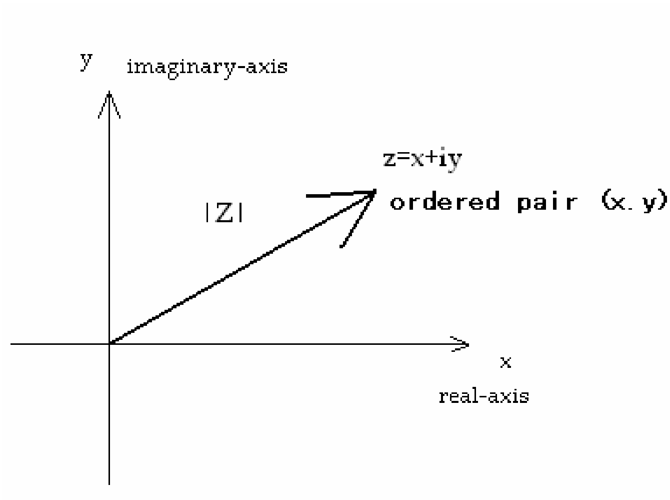
（2） conjugate of z = x + i y 共軛複數

$\bar{Z} = x - i\,y$

（3） Modulus or Absolute value

$$|Z| = \sqrt{x^2 + y^2} = \sqrt{Z + \bar{Z}}$$

（4） complex plane 〔z-plane〕



SO！ The geometric repressntation of complex numbers as points in the complex plane

（5） operation properties

$\bar{Z} = Z$

$\overline{Z_1 + Z_2} = \bar{Z}_1 + \bar{Z}_2$

:

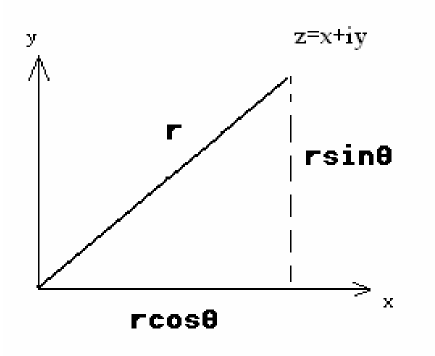
:

:

else

2. Polar form 複變數極座標形式

（1） polar form



＝＞ $Z = x + iy = r\cos\theta + r\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$Z = x + iy$$

$$= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right)$$

$r = |Z| = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}\frac{y}{x} = \arg(Z)$ ← argument 幅角

radian,positive,C.C.W

（2） principle argument Arg（Z）主輻角

The argument of a complex number $Z \neq 0$ in the internal $-\pi < \theta < \pi$

Ex. $Z = i$

arg（Z）= ?

Arg（Z）= ?

Sol：

$$Z = i = re^{i\theta} = |i|e^{i\theta} = e^{i\theta}$$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{0} = \frac{\pi}{2} \pm 2n\pi$ （n= 0.1.2.……）

so~ arg（Z）$= \theta = \frac{\pi}{2} \pm 2n\pi$ n= 0.1.2.……

Arg（Z）$= \frac{\pi}{2}$

（3） power of Z

$$Z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

ex. Z=1+i , $Z^{10} = ?$

sol： $r = |Z| = \sqrt{2}$, $\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{1}$

（4） DeMoivre's Formula

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta$ ｜Z｜= r = 1

This formula can be use for expressing $\cos n\theta$, $\sin n\theta$ in terms of $\cos\theta$ & $\sin\theta$

ex. $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$$
\begin{aligned}
&(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) \\
&= (\cos^2\theta - \sin^2\theta) + i2\cos\theta\sin\theta \\
&= (2\cos^2\theta - 1) + i2\cos\theta\sin\theta
\end{aligned}
$$

(5) Roots

Real numbers $X^3 = 1$

Roots: $(x-1)(x^2+x+1) = 0$

x=1 , $x = \dfrac{-1+\sqrt{3}i}{2}$

complex number

$w^n = z$

w : *roots* of z

let

$w = R(\cos\phi + i\sin\phi)$

$z = r(\cos\phi + i\sin\phi)$

$\therefore R^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$

or

$R^n e^{in\phi} = re^{i\theta}$

$\therefore R^n = r$

$n\phi = \theta + 2k\pi \quad (k = 0,1,2,3......, n = 1)$

$\mathrm{R} = \mathrm{r}^{1/n}$

$\phi = \dfrac{\phi + 2\mathrm{k}\pi}{\mathrm{n}} \qquad \mathrm{k} = 0,1,2,.............\mathrm{n}-1$

$\therefore \mathrm{w} = \mathrm{r}^{1/n}[\cos(\dfrac{\theta + 2k\pi}{n}) + i\sin(\dfrac{\theta + 2k\pi}{n})] \qquad \mathrm{k} = 0,1,2......, \mathrm{n}-1$

For k≥n , we obtain the same roots become the sine And cosine are 2π - periodic.

principle r^{th} root (k = 0)

$\mathrm{w} = \mathrm{r}^{1/n}[\cos\dfrac{\theta}{n} + i\sin\dfrac{\theta}{n}]$

Ex. $\sqrt[3]{z}$, z = i

Sol. Z=i

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\therefore r = 1 \ , \theta = \text{Arg(z)} = \frac{\pi}{2}$$

Ex. $\sqrt[3]{Z}$, Z = i

Sol: Z = i

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\therefore \quad r = 1 \ , \ \theta = Arg(Z) = \frac{\pi}{2}$$

2. $w_n = Z^{\frac{1}{3}}$ (n = 3)

$$= (1)^{\frac{1}{3}}[\cos(\frac{\frac{\pi}{2} + 2k\pi}{3}) + i\sin(\frac{\frac{\pi}{2} + 2k\pi}{3})]$$

k = 0

$$w_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

k = 1

$$w_1 = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

k = 2

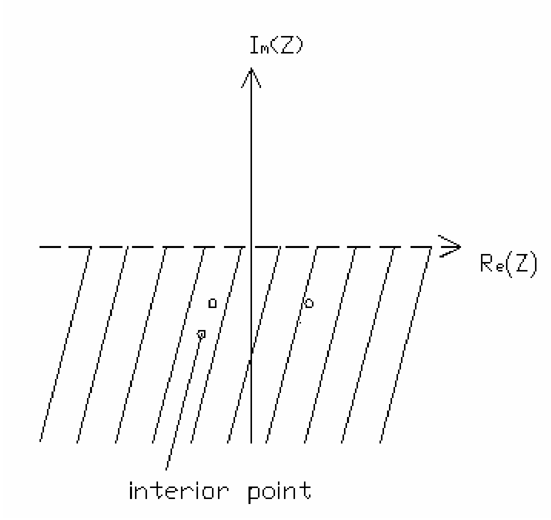
$$w_2 = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi = i$$

3. Sets of points in complex plane

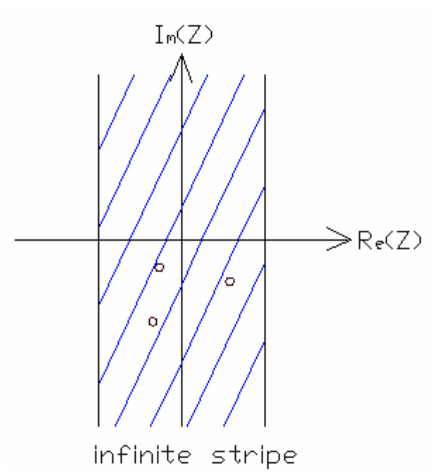
(1) open set

Every point in the set is an interior point .

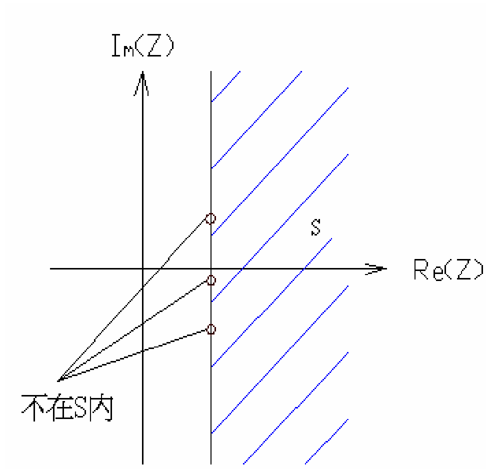
Ex. $I_m(Z) < 0$ (open set)



Ex. $-1 < R_e(Z) < 1$



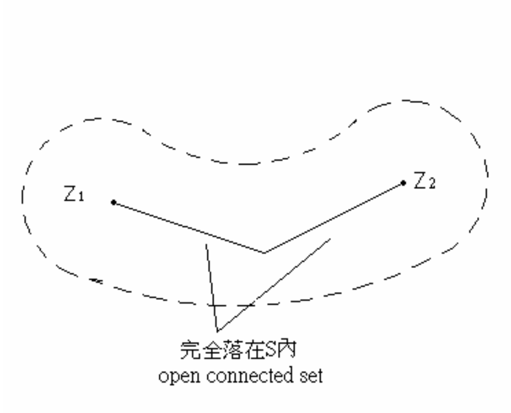
Ex. $R_e(Z) \geq 1$ (not open set)

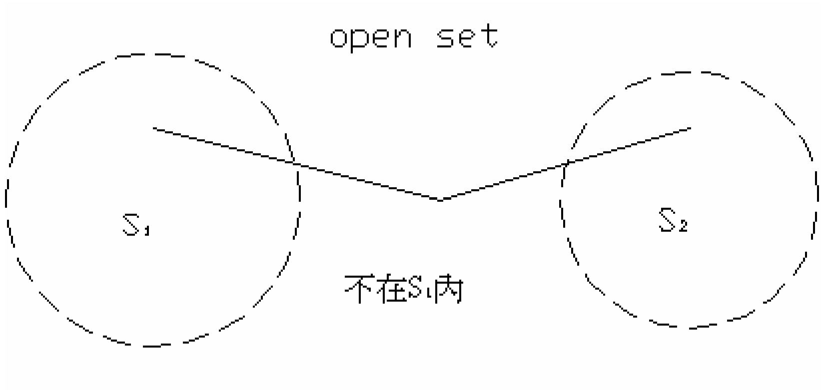


A. open connected set (Domain)

An open set in which any pain of points Z_1 and Z_2 can be connects by a polygonal line that lies entirely in the set .

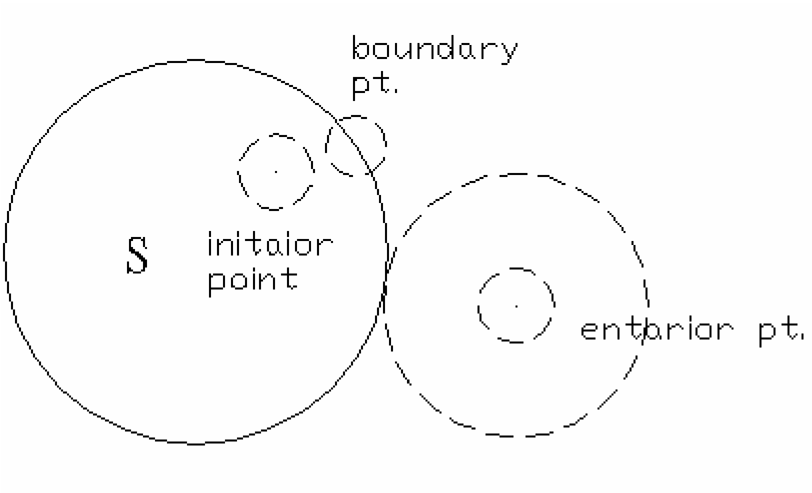
Ex.





(2) boundary

The set of all boundary points of set s



(3) Closed Region

A region contains all its boundary.

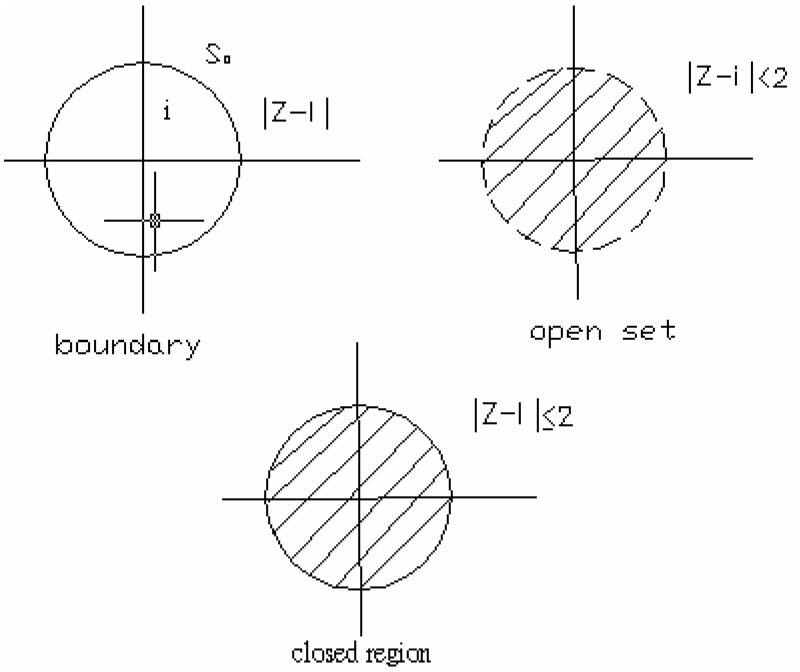
EX. $S_0 = \{Z \| z - i| = 2\}$

$S_1 = \{Z \| z - i| < 2\}$

$S_2 = \{Z \| z - i| \le 2\}$

Sol:

$|Z - i| = \sqrt{x^2 + (y-1)^2} = 2$

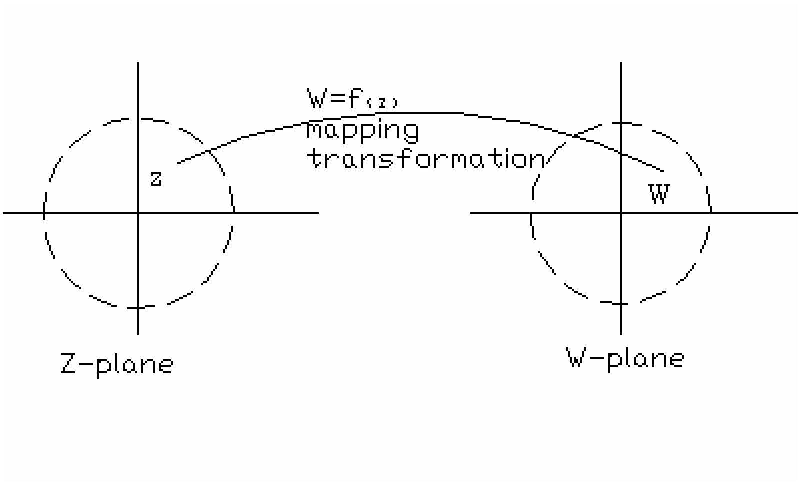


4. Complex function

(1) Complex function

Function of a complex variable.

$W = f(z) = u(x, y) + iv(x, y)$



EX.

Complex variable Re(z)=1

Complex function f(z)= Z^2

Sol:

$W = f(x) = Z^2 = (x+iy)^2 = (x^2 - y^2) + i2xy$

$\text{Re}(z) = 1$

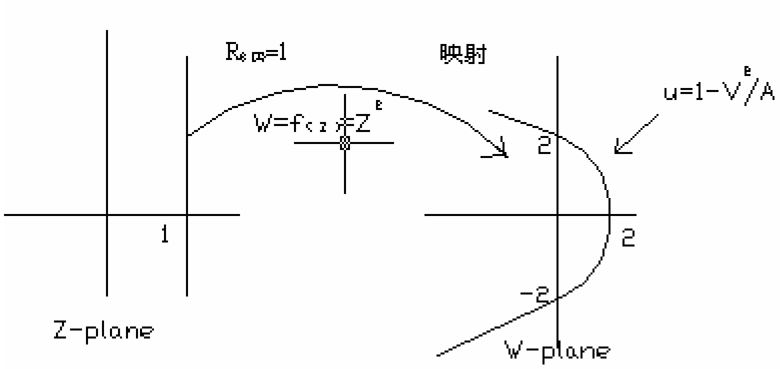
$\therefore Z = 1 + iy$

$\therefore W = (1-y^2) + i2y = u(x,y) + iv(x,y)$

$u = 1 - y^2$

$v = 2y$

$u = 1 - (\frac{v}{2})^2 = 1 - \frac{v^2}{4}$ ………W-plane



(2) Limit

Complex function f(z) is said to posses a limit at Z,

$\lim_{Z \to Z_0} f(Z) = L$ is for each $\varepsilon > 0$,such that $|f(z) - L| < \varepsilon$

wherever $0 < |Z - Z_0| < \varepsilon$

NOTE: $\lim_{Z \to Z_0} f(Z)$ exists that means f(z) approaches L as the point z approaches Z_0 from any direction.

(3) Continuity at a point

A function f(z) is continuous at a point Z_0 if $\lim_{Z \to Z_0} f(Z) = f(Z_0)$

(4) Derivative 導數

The derivative of f(z) at Z_0 is $\lim_{\Delta Z \to 0} \frac{f(Z_0 - \Delta Z) - f(Z_0)}{\Delta Z}$ $f'(Z_0) =$ provid this limit exists.

(5) Differentiable 可微分的

If the limit $\lim_{\Delta Z \to 0} \frac{f(Z_0 - \Delta Z) - f(Z_0)}{\Delta Z}$ or $\lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0}$ exists, the function $f(Z)$ is differentiable at Z_0

Ex

$f(Z) = Z^2$ differentiable ?

Sol:

$$\lim_{\Delta Z \to 0} \frac{f(Z_0 - \Delta Z) - f(Z_0)}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{(Z_0 + \Delta Z)^2 - Z_0{}^2}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{2Z_0 \Delta Z + \Delta Z^2}{\Delta Z}$$

$$= \lim_{\Delta Z \to 0} (2Z_0 + \Delta Z) = 2Z_0$$ (exits for all Z)

$\therefore$ $f(Z) = Z^2$ is differentiable on the entire complex plane

Ex. $f(Z) = \overline{Z}$ differentiable?

Sol:

$$\lim_{\Delta Z \to 0} \frac{f(Z_0 + \Delta Z) - f(Z_0)}{\Delta Z}$$

if $\Delta Z \to 0$ along real axis then $\Delta y = 0$

$$\Delta Z = \Delta x = \overline{\Delta Z}$$

$$\therefore \lim_{\Delta Z \to 0} \frac{\overline{\Delta Z}}{\Delta Z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x}{\Delta x} = 1$$

if $\Delta Z \to 0$ along imaginary axis then $\Delta x = 0$

$$\Delta Z = i\Delta y, \overline{\Delta Z} = i\Delta y$$

$$\therefore \lim_{\Delta Z \to 0} \frac{\overline{\Delta Z}}{\Delta Z} = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{-i\Delta y}{i\Delta y} = -1$$

thus $\lim_{\Delta Z \to 0} \frac{\overline{\Delta Z}}{\Delta Z}$ doesn't exist all Z

$$\therefore f(Z) = \overline{Z}\_is\_not\_differentiable\_at\_any\_Z$$

(6) Analytic at a point Z_0

A complex function $f(Z)$ is said to be analytic at a point Z_0 if $f(Z)$ is differentiable at Z_0 and at every point in some neighborhood at Z_0,0<|Z- Z_0|< δ

Ex. $f(Z)=\left|Z^2\right|$ is differentiable ? analytic?

Sol:

$f(Z)=\left|Z^2\right|=Z\overline{Z}$ Look at

$$\lim_{\Delta Z\to 0}\frac{f(Z+\Delta Z)-f(\Delta Z)}{\Delta Z}=\lim_{\Delta Z\to 0}\frac{(Z+\Delta Z)\sqrt{\overline{(Z+\Delta Z)}}-Z\overline{Z}}{\Delta Z}$$

$$=\lim_{\Delta Z\to 0}\frac{Z\overline{\Delta Z}+\overline{Z}\Delta Z+\Delta Z\overline{\Delta Z}}{\Delta Z}=\lim_{\Delta Z\to 0}(Z\frac{\overline{\Delta Z}}{\Delta Z}+\overline{Z})$$

if $\Delta Z\to 0$ along real axis, then

$$\lim_{\Delta Z\to 0}(Z\frac{\overline{\Delta Z}}{\Delta Z}+\overline{Z})=\lim_{\substack{\Delta y\to 0\\ \Delta x=0}}(Z\frac{\overline{\Delta x}}{\Delta x}+\overline{Z})=-Z+\overline{Z}$$

If $\Delta Z\to 0$ along imaginary

Then $\lim_{\Delta z\to 0}\left(Z\frac{\overline{\Delta Z}}{\Delta Z}+\overline{Z}\right)=\lim_{\substack{\Delta x\to 0\\ \Delta y=0}}\left(Z\frac{\overline{i\Delta y}}{i\Delta y}+\overline{Z}\right)=-Z+\overline{Z}$ $\Delta Z=0$ 極限值相等

$$\therefore Z+\overline{Z}=-Z+\overline{Z}\Rightarrow\therefore Z=0$$

$$\therefore f(Z)=\left|Z\right|^2$$

$\therefore f(Z)=\left|Z\right|^2$ is differentiable at $Z=0$ only but is not differentiable anywhere also

$\Rightarrow f(Z)=\left|Z\right|^2$ is no where analytic

(7) entire function 完全函數

A complex function is analytic at every point Z

Ex.

$$\begin{cases} polyno\min al\_function\_f(Z)=\left|Z\right|^2 \\ \exp rential\_function\_f(Z)=e^z \end{cases}$$

5. Cauchy-Riemann Equation

(1) Cauchy-Riemann equation

if $f(Z)-u(x,y)+iv(x,y)$ is continuous in neighborhord of Z and differentiable at Z itself, then first order partial derivatives of $u(x,y)$ and $v(x,y)$ exists and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$[proof]$ $f(Z)$ is differentiable at $Z \Rightarrow f'(Z) = \lim_{\Delta z\to 0}\frac{f(Z+\Delta Z)-F(Z)}{\Delta Z}$

$$\therefore \lim_{\Delta z\to 0}\frac{f(Z+\Delta Z)-F(Z)}{\Delta Z} = \lim_{\Delta z\to 0}\frac{[u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)]-[u(x,y)+iv(x,y)]}{\Delta x+i\Delta y}$$

$\Delta Z \to 0$ along real axis

$$\therefore \lim_{\Delta z\to 0}\frac{f(Z+\Delta Z)-F(Z)}{\Delta Z} = \lim_{\substack{\Delta x\to 0\\ \Delta y=0}}\frac{[u(x+\Delta x, y)+iv(x+\Delta x, y)]-[u(x,y)+iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{[u(x+\Delta x, y)-u(x,y)]}{\Delta x}+i\lim_{\Delta x\to 0}\frac{[v(x+\Delta x, y)-v(x,y)]}{\Delta x} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

if $\Delta z \to 0$ along imaginary axis ,then

$\Delta z = i\Delta y$, $\Delta x = 0$

$$\therefore \quad f'(z) = \lim_{\substack{\Delta y\to 0\\ \Delta x=0}}\left\{\frac{[u(x, y+\Delta y)+iv(x, y+\Delta y)]-[u(x,y)+iv(x,y)]}{i\Delta y}\right\}$$

$$= \lim_{\Delta y\to 0}\left\{\frac{v(x, y+\Delta y)-v(x,y)}{\Delta y}-i\frac{u(x, y+\Delta y)-u(x,y)}{\Delta y}\right\}$$

$$= \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$ … C-R Eq.

C-R is Recessary condition not sufficient for an analytic equation

無之必然

有之未必然

Ex: $f(z) = \bar{z}$ differentiable ?

(sol): use Cauchy-Rieman equation

$$f(z) = \bar{z} = x - iy = u(x,y) + iv(x,y)$$

$$u(x,y) = x, ... v(x,y) = -y$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = 1 \neq \dfrac{\partial v}{\partial y} = -1 \\ \dfrac{\partial u}{\partial x} = 0 = -\dfrac{\partial u}{\partial y} \end{cases}$$

⇨ C-R Eq. is not satisfied

⇨ $f(z) = \bar{z}$ not analytic at any pt. z

(2) Sufficient & Necessary condition for analyticity

Let $f(z) = u(x,y) + iv(x,y)$,If u, u_x, u_y, v, v_x, v_y are continuous in domain D ,and Cauchy-Rieman equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied ,then $f(z)$ is analytic in D

Ex: $f(z) = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$ Analytic?

$$\begin{cases} u(x,y) = \dfrac{x}{x^2 + y^2} \\ v(x,y) = \dfrac{-y}{x^2 + y^2} \end{cases}$$

when $x^2 + y^2 = 0$,u & v are not continuons

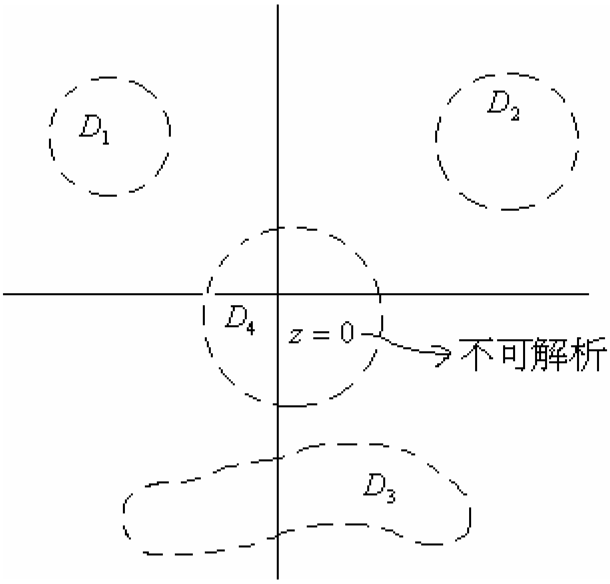
C-R Eqs.

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = \frac{-\partial v}{\partial x}$$

$\therefore\ f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$ is not analytic in any domain containing z=0

$x = 0, y = 0 \ldots \because x^2 + y^2 = 0$



$f(z)$ is not analytic in D_4 but is analytic in D_1, D_2 and D_3.

6 . Exponential and logarithmic function

(1) exponential function $e^z, (\exp(z))$

$$e^z = e^{x+iy} = e^x e^{iy}$$
$$= e^x(\cos y + i\sin y)$$
$$(e^z)^1 = e^z$$

$$e^{z_1 z_2} = e^{z_1 + z_2} \qquad \frac{e^{z_1}}{e^{z_2}} = e^{z_1 + z_2}$$

(2) Complex logarithmic function $(z \neq 0)$

$$w = \ln z \overset{Inverse}{\Leftrightarrow} z = e^w$$

$$w = u + iv \qquad z = x + iy$$

$$\therefore x + iy = e^{u+iv} = e^{u}(\cos v + i\sin v)$$

$$\begin{cases} x = e^{u}\cos v \\ y = e^{u}\sin v \end{cases}$$

$$\therefore x^2 + y^2 = (e^{u})^2, \therefore e^{u} = \sqrt{x^2 + y^2}$$

$$u = \ln\sqrt{x^2 + y^2} = \ln|z|$$

$$\frac{y}{x} = \frac{\sin v}{\cos v} = \tan v$$

$$v = \tan^{-1}(\frac{y}{x}) = \theta = \arg(z)$$

$$\theta + 2\pi \qquad n = 0, \pm 1, \pm 2 \ldots\ldots$$

$$\therefore \ln z = \ln|z| + i(\theta + 2n\pi)$$

multiple-valued function

principle value of logarithmic fn.

$\ln z = \ln|z| + iArg(z) \ldots$ single-valued fn.

$$-\pi < Arg(z) < \pi$$