IV Integral Transformation method

-A integral transformation is a transformation that produces from a given function which depends on a different variable and appears in the from of an integral.

EX: Laplace transform

$$F(x) = \int_0^\infty e^{-st} f(t)dt = L[f(x)]$$

When to apply Integral Transformation method?

(1) P.D.E is nonhomogeneous

(2) Boundary Condition and time dependent

(3) Semi-infinite(0<x<∞) or infinite (-∞<x<∞) spatial domain

1. Solution by Laplace transform

- Laplace transform of a function of two variable u(x,t) with respect to the variable t by

$$L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)dt = U(x,s)$$

$$L[\frac{\partial u}{\partial t}] = sL[u] - u(x,0)$$
$$= U - u(x,0)$$

$$L[\frac{\partial^2 u}{\partial t^2}] = s^2 U(x,s) - su(x,0) - u_t(x,0)$$

$$L[\frac{\partial^2 u}{\partial x^2}] = \int_0^\infty e^{-st} \frac{\partial^2 u}{\partial x^2} dt$$
$$= \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u(x,t)dt$$
$$= \frac{d^2}{dx^2} U(x,s)$$

=> Ording D.E. for U(x, s)

=> Solution for U(x, s)

=> Obtain solution u(x, t) by Inverse Laplace transform

Example：

Semi-infinite string

Wave equation

$$\frac{\partial^2 w}{\partial t^2} = C^2 \frac{\partial^2 w}{\partial x^2} \quad 0 < x < \infty, t > 0$$

BCs

$$\begin{cases} w(0,t) = f(t) & t > 0 \\ \lim_{x\to\infty} w(x,t) = 0 \end{cases}$$

ICs

$$w(x,0)=0 \quad 0<x<\infty$$

$$\left.\frac{\partial\infty}{\partial t}\right|_{t=0}=0$$

Sol： Use Laplace transform w.r.t. t

$$L[\frac{\partial^2 w}{\partial t^2}]=s^2 w(x,s)-sw(x,0)-wt(x,0)$$

$$=C^2 \mathrm{L}[\frac{\partial^2 w}{\partial x^2}]$$

$$=\mathrm{C}^2\,\frac{d^2 L[w]}{dx^2}=C^2\,\frac{d^2 w[x,s]}{dx^2}$$

$$\therefore \frac{d^2 w}{dx^2}-\frac{s^2}{c^2}w=0$$

O.D.E for w(x, s)

General solution of w(x, s)

$$w(x,s)=A(s)e^{\frac{s}{c}x}+B(s)e^{-s/c\,x}$$

Boundary condition

$$L[w(0,t)]=w(0,s)$$

$$=\mathrm{L[f(t)]=F(s)}$$

$$\mathrm{L}[\lim_{x\to\infty}\mathrm{w(w,t)}]=\lim_{x\to\infty}w(x,s)=0$$

$$\lim_{x\to\infty}w(x,s)=\lim_{x\to\infty}[A(s)e^{\frac{s}{c}x}+B(s)e^{-s/c\,x}]=0$$

$$\therefore A(s)=0$$

$$\mathrm{W(0,s)=B(s)=F(s)}$$

$$\therefore \mathrm{w(x,s)=F(s)e^{-s/c\,x}}$$

From Second-shifting theorem

$$L[f(t-a)u(t-a)\}=e^{-as}F(s)$$

$$L[f(t)]=F(s)$$

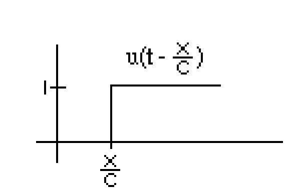
$$\therefore L^{-1}[e^{-as}F(s)=f(t-a)u(t-a)]$$

$$L^{-1}[F(s)]=f(t)$$

$$\therefore \mathrm{w(x,t)=L^{-1}}[w(x,s)]$$

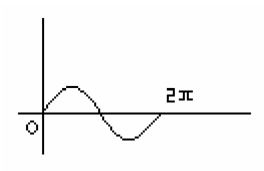
$$=\mathrm{L^{-1}}[F(s)e^{-s/c\,x}]\quad (a=\frac{-x}{c})$$

$$=f(t-\frac{x}{c})u(t-\frac{x}{c})$$



$$u(t-\frac{x}{c}) = \begin{cases} 0 & t < \frac{x}{c} \\ 1 & t \geq \frac{x}{c} \end{cases}$$

$$if \quad f(t) = \begin{cases} \sin t & 0 \leq t \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$



$$\therefore w(x,t) = \sin(t - \frac{x}{c}) u(t - \frac{x}{c})$$

$$= \begin{cases} \sin(t - \frac{x}{c}) & 2\pi + \frac{x}{c} > t > \frac{x}{c} \\ 0 & \text{otherwise} \end{cases}$$

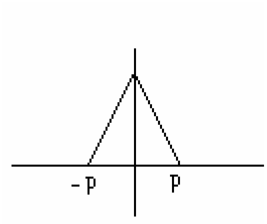
2. Solutions by Fourier Transform

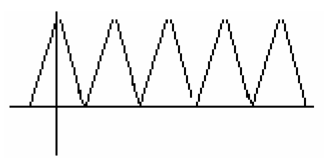
(1) Fourier Integral：

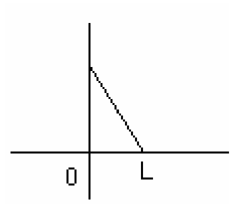
If f(x) is piecewise continuous in every finite interval and has a right-hand

Derivative and a left-hand derivative at every point and if $\int_{-\infty}^{\infty} |f(x)| dx$

Exist, then f(x) can be represented by a Fourier Integral







$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x] d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos\alpha x dx$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin\alpha x dx$$

Fourier Series：

f(x) defined on finite interval (-p, p) or (0,L) or Periodic function

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{p}+B_n\sin\frac{n\pi x}{p})\ \text{ on [-p, p]}$$

Fourier Integral：

f(x) is nonperiodic function defined on $(-\infty,\infty)$ or $(0,\infty)$

From Fourier series to Fourier Integral：

If f(x) defined on [-p, p]

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{p}+B_n\sin\frac{n\pi x}{p})$$

$$=\frac{1}{2p}\int_{-p}^{p}f(x)dx+\frac{1}{p}\sum_{n-1}^{\infty}[\int_{-p}^{p}f(x)\cos\frac{n\pi x}{p}dx\cos\frac{n\pi x}{p}+\int_{-p}^{p}f(x)\sin\frac{n\pi x}{p}dx\sin\frac{n\pi x}{p}]$$

Let

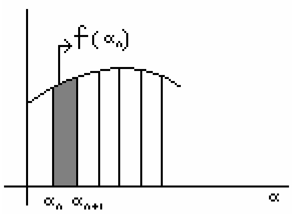
$$\alpha_n=\frac{n\pi}{p}$$

$$\therefore\Delta\alpha=\alpha_{\mathrm{n+1}}-\alpha_n=\frac{\pi}{p}$$

$$\therefore\ f(x)\frac{1}{2\pi}[\int_{-p}^{p}f(x)dx]\Delta\alpha+\frac{1}{\pi}\sum_{n-1}^{\infty}[\int_{-p}^{p}f(x)\cos\alpha_n xdx\cos\alpha_n x+\int_{-p}^{p}f(x)\sin\alpha_n xdx\sin\alpha_n x]\Delta\alpha$$

Now let $p\to\infty\quad\therefore\Delta\alpha=\frac{\pi}{\mathrm{p}}\to 0$

$$\therefore\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty}F(\alpha_n)\Delta\alpha=\int_0^{\infty}F(\alpha)d\alpha$$



If $\int_{-\infty}^{\infty}|f(x)|dx$ exists

$$\therefore\lim_{\Delta\alpha\to 0}\frac{1}{2\pi}\left(\int_{-\infty}^{\infty}f(t)dt\right)\Delta\alpha=0$$

$$\therefore\ f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty}f(x)\cos\alpha xdx\right)\cos\alpha x+\left(\int_{-\infty}^{\infty}f(x)\sin\alpha xdx\right)\sin\alpha x\right]d\alpha$$

$$=\frac{1}{\pi}\int_0^{\infty}[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]d\alpha\ \ldots\text{Fourier integral of } f(x)\text{ on}(-\infty,\infty)$$

Here

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$$

$$B(\alpha)=\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$$

Fourier cosine & sine integrals(非週期定義在$(-\infty,\infty)$)

(i)Fourier cosine integral

$f(x)$ is even function on $(-\infty,\infty)$

$$f(x)=\frac{2}{\pi}\int_{-\infty}^{\infty} A(\alpha)\cos\alpha x dx$$

$$A(\alpha)=\int_{0}^{\infty} f(x)\cos\alpha x dx$$

(ii)Fourier sine integral

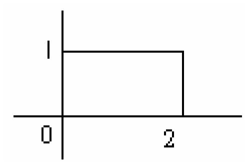
$f(x)$ is odd function on $(-\infty,\infty)$

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty} B(\alpha)\sin\alpha x dx$$

$$B(\alpha)=\int_{0}^{\infty} f(x)\sin\alpha x dx$$

Ex. Find Fourier integral of $f(x)$

$$f(x)=\begin{cases}0 & x<0\\ 1 & 0<x<2\\ 0 & x>2\end{cases}$$



sol. Fourier Integral of $f(x)$

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]d\alpha$$

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos\alpha x dx=\int_{0}^{2}\cos\alpha x dx=\frac{\sin\alpha x}{\alpha}\Big|_{0}^{2}=\frac{\sin 2\alpha}{\alpha}$$

$$B(\alpha)=\int_{0}^{2}\sin\alpha x dx=\frac{1-\cos 2\alpha}{\alpha}$$

$$\therefore\ f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\sin 2\alpha}{\alpha}\cos\alpha x+\frac{(1-\cos 2\alpha)}{\alpha}\sin\alpha x\right]d\alpha$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\alpha\cos\alpha(x-1)}{\alpha}d\alpha$$

Take x=1

$$f(1)=1=\frac{2}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha$$

$$\therefore \int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha=\frac{\pi}{2}\ \#$$

Complex form of Fourier Integral

$$f(x)=\frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]\,d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)[\cos\alpha t\cos\alpha x+\sin\alpha t\sin\alpha x]\,dt\,d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\cos\alpha(x-t)\,dt\,d\alpha \qquad (even\ \ fn.\ of\ \ \alpha)$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)[\cos\alpha(x-t)+i\sin\alpha(x-t)]\,dt\,d\alpha \qquad \left(\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\sin\alpha(x-t)\,dt\,d\alpha=0\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)e^{-i\alpha(x-t)}\,dt\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty f(t)e^{-i\alpha t}\,dt\right]e^{i\alpha x}\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty c(\alpha)e^{i\alpha x}\,dx$$

$$\therefore\ f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty c(\alpha)e^{i\alpha x}\,d\alpha$$

$$c(\alpha)=\int_{-\infty}^\infty f(x)e^{-i\alpha x}\,dx$$

(2)Fourier Transform

Integral, transform occurs in transform pair

$$f(x)\ \underset{Inverse}{\overset{\int}{\rightleftarrows}}\ F(\alpha)$$

$$F(\alpha)=\int_a^b f(x)k(\alpha,x)\,dx$$

$$f(x)=\int_c^d F(\alpha)H(\alpha,x)\,d\alpha$$

Here $k(\alpha,x)$ and $H(\alpha,x)$ are called “Kernels” of transform.

Ex.Laplace transform pairs Laplace transform $F(s)=\int_0^\infty f(t)e^{-st}\,dt$

Inverse Laplace transform $f(t)\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}F(s)e^{st}ds$

・use of complex variable

・complex contour integral

A. Fourier Transform Pairs

source: Fourier Integral

(i) Fourier transform $F(f(x))=\int_{-\infty}^{\infty}f(x)e^{idx} = \hat{f}(\alpha)$

Inverse Fourier transform $F^{-1}(\hat{f}(\alpha))\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\alpha)e^{i\alpha x}d\alpha = f(x)$

(ii) Fourier Sine transform $F_s(f(x))=\int_0^{\infty}f(x)\sin xdx=\hat{f}_s(\alpha)$

Inverse Fourier Sine transform $F_s^{-1}(\hat{f}_s(\alpha))=\frac{2}{\pi}\int_0^{\infty}\hat{f}_s(\alpha)\sin\alpha x d\alpha = f(x)$

(iii) Fourier cosine transform $F_c(F(x))=\int_0^{\infty}f(x)\cos\alpha x dx=\hat{f}_c(\alpha)$

Inverse Fourier Cosine transform $F_C^{-1}(\hat{f}_c(\alpha))\cos\alpha x d\alpha = f(x)$

B. Existence Conditions

(i) *f(x)* and $f'(x)$ is piecewise continuous on every finite interval.

(ii) *fx()* is absolutely integrable on (-∞, ∞),namely

$\int_{-\infty}^{\infty}|f(x)|dx$ exists

Note:F(1), $F_s(1)$,and $F_c(1)$ not exist.

C. Operational properties

(i) liner operator

$$F(af(x)+bg(x))=aF(f(x))+bF(g(x))$$
$$F_s(af(x)+bg(x))=aF_s(f(x))+bF_s(g(x))$$
$$F_c(af(x)+bg(x))=aF_c(f(x))+bF_c(g(x))$$

(ii) Transform of Derivatives

(a) Fourier transform of derivatives of f(x)

$$F(f'(x))=\int_{-\infty}^{\infty}f'(x)e^{-i\alpha x}dx=\int_{-\infty}^{\infty}e^{-i\alpha x}df(x)= f(x)e^{-i\alpha x}\Big|_{-\infty}^{\infty}+i\alpha\int_{-\infty}^{\infty}f(x)e^{-i\alpha x}dx=i\alpha F(f(x))$$

$\therefore F(f'(x))=i\alpha F(f(x))$

$F(f''(x))=i\alpha F(f'(x))=(i\alpha)^2F(f(x))$

(b) Fourier Cosine and Sine transform

$$F_c(f^{'}(x)) = \int_0^{\infty} f(x)\cos\alpha x dx = f(x)\cos\alpha x\Big|_0^{\infty} + \alpha\int_0^{\infty} f(x)\sin\alpha x dx$$

$$F_s(f^{'}(x)) = \int_0^{\infty} f^{'}(x)\sin\alpha x dx = f(x)\sin\alpha x\Big|_0^{\infty} - \alpha\int_0^{\infty} f(x)\cos\alpha x dx$$

Similanly $\therefore F_s(f^{'}(x)) = -\alpha F_c(f(x))$

$$\therefore F_c(f^{''}(x)) = -f^{'}(0) + \alpha F_s(f^{'}(x)) = -f^{'}(0) + \alpha(-\alpha F_c(f(x)))$$

$$\therefore F_c(f^{''}(x)) = -f^{'}(x) - \alpha^2 F_c(f(x))$$

Similanly $F_s(f^{''}(x)) = \alpha f(0) - \alpha^2 F_s(f(x))$

(3)Solutions by Fourier Transform

When to use Fourier transform

(i) Domain (-∞, ∞) for Fourier transform

(ii) Domain [-∞, ∞) for Fourier cosine or Sine transform

A. Heat problem on infinite rod

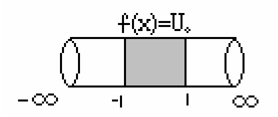
$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad -\infty < x < \infty,\ t > 0$$

I.C.

u(x,0) = f(x)

$$= \begin{cases} U_0 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

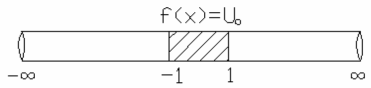
U(x, t) , $\dfrac{\partial u(x,t)}{\partial t} \to 0$ as $x = \pm\infty$



Ex :

Heat problem on infinte rod , $C^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ $\quad -\infty < x < \infty$, t > 0

I.C u(x,0)=f(x)= $\begin{cases} U_o & |x| < 1 \\ 0 & |x| > 1 \end{cases}$, u(x,t) , $\dfrac{\partial u(x,t)}{\partial t} \to 0$ as $x = \pm\infty$



Sol :

Use Fourier transform on variable x

$$F(u(x,t)) = \int_{-\infty}^{\infty} u(x,t)e^{-idx}dx = \hat{u}(\alpha,t) \quad , \quad \therefore F\left(c^2 \frac{\partial^2 u}{\partial x^2}\right) = F\left(\frac{\partial u}{\partial t}\right)$$

$$\Rightarrow -c^2\alpha^2 F(u(x,t)) = \frac{d}{dt}F(u(x,t))$$

$$\Rightarrow -c^2\alpha^2 \hat{u}(\alpha,t) = \frac{d\hat{u}(\partial,t)}{dt}$$

or

$$\frac{d\hat{u}(\alpha,t)}{dt} + c^2\alpha^2 \hat{u}(\alpha,t) = 0$$

$\cdots 1^{st}$ order O.D.E with t as independent variable

$\therefore$ general solution fn $\hat{u}(\alpha,t)$

$$\hat{u}(\partial,t) = A(\alpha)e^{-c^2\alpha^2 t}$$

$A(\alpha) = ?$

Fourier transform of initial condition

$$F(u(x,0)) = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$$

$$\hat{u}(\alpha,0) = \hat{f}(\alpha)$$

$$\therefore I.C.\hat{u}(\alpha,0) = \hat{f}(\alpha)$$

$$\therefore \hat{u}(\partial,0) = A(\alpha) = \hat{f}(\alpha)$$

Hence

$$\hat{u}(\alpha,t) = \hat{f}(\alpha)e^{-c^2\alpha^2 t}$$

Form Inverse Fourier transform , we have

$$u(x,t) = F^{-1}\left(\hat{u}(\alpha,t)\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\alpha,t)e^{i\alpha x}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)e^{-i^2\alpha^2 t}e^{i\alpha x}dx$$

Here

$$f(x) = \begin{cases} U_o & |x| \le 1 \\ 0 & x > 1 \end{cases}$$

There

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$$

$$= \int_{-1}^{1} U_o e^{-i\alpha x}dx = -\frac{U_o e^{-i\alpha x}}{i\alpha} \Big|_{-1}^{1}$$

$$= U_o \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha}$$

$$= 2\frac{Sin\alpha}{\alpha}U_o$$

$$\therefore u(x,t) = \frac{U_o}{\pi}\int_{-\infty}^{\infty} \frac{Sin\alpha}{\alpha} e^{-c^2\alpha^2 t}e^{-i\alpha x}dx$$

Euler formula

$$e^{-i\alpha x} = Cos\alpha x - iSinx$$

$$\therefore u(x,t) = \frac{U_o}{\pi}\int_{-\infty}^{\infty} \frac{SinCos\alpha x}{\alpha} e^{-c^2\alpha^2 t}d\alpha - i\frac{U_o}{\pi}\int_{-\infty}^{\infty} \frac{Sin\alpha Sin\alpha x}{\alpha} e^{-c^2\alpha^2 t}d\alpha$$

$$\therefore u(x,t) = \frac{U_o}{\pi}\int_{-\infty}^{\infty} \frac{SinCos\alpha x}{\alpha} e^{-c^2\alpha^2 t}d\alpha$$

Ex ：

Heat eqn on semi-infinite rod , $C^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ $0 \le x < \infty$, t>0

I.C. $u(x,0) = f(x)$ $0 < x < \infty$

B.C. $u(0,t) = 0$

Sol :

$\because BC$ $u(0,t) = 0$

$\therefore$ use Fourier series transform on variable x

$$c^2 F_s(u_{xx}) = -c^2\alpha^2 F_s(u) + c^2\alpha u(0,t)$$

$$= -c^2\alpha^2 \hat{u}_s(\alpha, t)$$

$$F_s\left(\frac{\partial u}{\partial \tau}\right) = \frac{d}{dt} F_s(u) = \frac{d\hat{u}_s(\alpha, t)}{\partial t}$$

$$\therefore \frac{d\hat{u}_s}{dt} + c^2\alpha^2 \hat{u}_s = 0$$

Solution $\hat{u}_s$ is $\hat{u}_s(\alpha, t) = A(\alpha)e^{-c^2\alpha^2 t}$

I.C. $F_s(u(x,0)) = \hat{u}_s(\alpha, 0) = \int_0^\infty f(x) Sin\alpha x dx = \hat{f}_s(\alpha)$

$$\therefore A(\alpha) = \hat{f}_s(\alpha)$$

$$\therefore \hat{u}_s(\alpha, t) = \hat{f}_s(\alpha)e^{-c^2\alpha^2 t}$$

By Inverse Fourier sine transform , we have

$$u(x,t) = F_s^{-1}\left(\hat{u}_s(\alpha, t)\right)$$

$$= \frac{2}{\pi}\int_0^\infty \hat{u}_s(\alpha, t)\sin\alpha x d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \hat{f}_s(\alpha)e^{-c^2\alpha^2 t}\sin\alpha x dx$$

Note :

If BC at x=0 is $\frac{\partial u}{\partial x}\Big|_{x=0} = 0$ then cosine the Fourier cosine transform .

H.W.

Stesdy – state temperature of semi-infinite plate .

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < \pi \quad , y > 0$$

$$\frac{\partial u}{\partial y}\Big|_{y=0} = 0 \qquad 0 < x < \pi$$