Ⅲ Partial Differential Equations

1. Basic concept

(1)Partial differential equation (P.D.E)

A. DE with two or more independent variable (x,y,z,t)

General form

F(x,y,z,t,u, u_x, u_y, u_z, v…… ,)≡0

u ,v,……=dependent variables

(2) Order

The order of higheat desivatives…

Ex. $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{order} = 2$$

(3) Degree

The higheat power of the higheat derivative

Ex.

$$\frac{\partial u}{\partial t} = c^2 (\frac{\partial^2 u}{\partial x^2}) \quad \text{1st degree p.d.e}$$

$$(\frac{\partial u}{\partial t})^2 = c^2 (\frac{\partial^2 u}{\partial x^2}) \text{1st degree p.d.e}$$

$$\frac{\partial u}{\partial t} = c^2 (\frac{\partial^2 u}{\partial x^2})^2 \quad \text{2nd degree p.d.e}$$

(4) Linear

A P.D.E is linear if it is of the first degree in the dependent variable and its partial derivative .

Ex.

$$A\frac{\partial^2 u}{\partial^2 x} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial^2 y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y)$$

where A,B,C,D,E,F are real constants .

$$2^{nd}\, order, 1^{st} - \deg ree, linear P.D.E.$$

(5) Homogenity

$$\begin{cases} G(x, y) = 0 & Homogeneous\ Eq. \\ G(x, y) = 0 & Non\ homogeneous\ Eq. \end{cases}$$

(6) Solution of a P.D.E.

A function has all derivatives within region R and satisfies the P.D.E. in the interior of R .

2. P.D.E. in Rectangular coordinates

(1) Separable P.D.E.

a. Linear Eq.

$$2^{nd} - order \quad linaer \quad P.D.E.$$

$$A\frac{\partial^2 u}{\partial^2 x} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial^2 y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y)$$

where A,B,C,D,E,F are real constants

$AX^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ 　　*圓錐曲線*

(a) Hyperbolic－Type P.D.E

If $B^2 - 4AC > 0$

vibrating system, wave equation

EX. $\Im\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

A=3 　B=0 　C=－1 　$B^2 - 4AC = 12 > 0$

(b) Parabolic－Type P.D.E

If $B^2 - 4AC = 0$

heat flow, diffusion process

EX. $\Im\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$

A=3 　B=0 　C=－1 　$B^2 - 4AC = 0$

(c) Elligtic－Type P.D.E

If $B^2 - 4AC < 0$

steady－state, phemomena

EX. $\Im\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$

A=3 　B=0 　C=－1 　$B^2 - 4AC = -4 < 0$

b. Superposition Principle

If $u_1, u_2 \cdots u_k$ are solution of a homogeneous linear P.D.E, then

$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$, $c_i = constant$ $i = 1,2,\cdots,k$

is also a solution of P.D.E.

c. Method of Separation of variable

Assume solution of P.D.E u(x,y)=X(x)Y(y) product method.

d. Example

Linear P.D.E

$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$ u(x,y)=?

Sol~

By method of Separation of variable, we assume

u(x,y)= $\overline{X}(x)\overline{Y}(y)$

$$\therefore \frac{\partial u}{\partial x} = \overline{X}'\overline{Y}(\overline{X}' = \frac{d\overline{X}}{dx}), \frac{\partial^2 y}{\partial x^2} = \overline{X}''\overline{Y}, \frac{\partial u}{\partial y} = \overline{XY}'(\overline{Y}' = \frac{d\overline{Y}}{dy})$$

$$\therefore \overline{X}''Y = 4\overline{XY}' \text{ or } \frac{\overline{X}''}{4\overline{X}} = \frac{\overline{Y}'}{\overline{Y}} = \text{constant } (\lambda^2 \text{ or } -\lambda)$$

Case 1 $\lambda^2 > 0$

$$\frac{X''}{4x} = \frac{\overline{Y}'}{Y} = \lambda^2$$

$$\therefore \begin{cases} x'' - 4\lambda^2 x = 0 \\ Y' - \lambda Y = 0 \end{cases} \text{ two O.D.E}$$

Solutions are $X(x) = C_1\cosh(2\lambda x) + C_2\sinh(2\lambda x)$

$$Y(y) = C_3\ell^{\lambda^2 y}$$

$$\therefore u(x,y) = X'(x)Y(y) = \ell^{\lambda^2 y}(A_1\cosh(2\lambda x) + B_1\sinh(2\lambda x))$$

Case 2 $\lambda^2 = 0$

two O.D.E

$$\begin{cases} X'' = 0 & X(x) = C_3 x + C_4 \\ Y' = 0 & Y(y) = C_5 \end{cases}$$

$$\therefore u(x,y) = A_2 x + B_2$$

Case 3 $\lambda^2 < 0$

two O.D.E

$$\begin{cases} X'' + 4\lambda^2 x = 0 \\ Y' + \lambda^2 Y = 0 \end{cases}$$

$$X(x) = C_6\cos(2\lambda x) + C_7\sin(2\lambda x)$$

$$Y(x) = C_8 e^{-x^2 y}$$

$$\therefore u(x,y) = e^{-\lambda^2 y}(A_3\cos(2\lambda x) + B_3\sin(2\lambda x))$$

NOTE:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = x \qquad \text{(NOT SEPARABLE)}$$

u(x,y)=?

(2) Classical Equation & problems

($\dot{1}$) Equation

A. Heat equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

B. Wave equation

$$\mathrm{d}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

C. Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \forall^2 u = 0$$

($\dot{2}$) Initial condition (t = 0)

u(x,y)=f(x)

and/or $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad 0<x<L$

($\dot{3}$) Boundary condition (t > 0)

A. Dirichlet condition , u

$$\text{Ex.}\begin{cases} u(o,t) = u_0(t) \\ u(L,t) = u_2(t) \end{cases}$$

B. Nuemann condition , $\frac{\partial u}{\partial n}$

$$\text{Ex.}\begin{cases} \left.\frac{\partial u}{\partial x}\right|_{x=0} = f(x) \\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = g(x) \end{cases}$$

C. Robin (churchill) condition

$$\frac{\partial u}{\partial n} = hu \quad \text{(mixed B.C)}$$

(iv). Modification of DE

Internal or external effect

$$\begin{cases} K\dfrac{\partial^2 u}{\partial x^2} + G(x,t,u) = \dfrac{\partial u}{\partial t} \\ a^2\dfrac{\partial^2 u}{\partial x^2} + F(x,t,u,u_t) = \dfrac{\partial^2 u}{\partial t^2} \end{cases}$$

Ex. $K\dfrac{\partial^2 u}{\partial x^2} - h(u-u_0) = \dfrac{\partial u}{\partial t}$

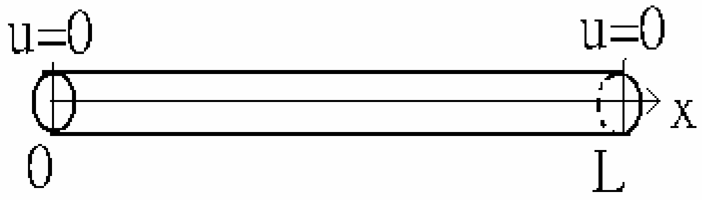
$$a^2\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + Ku$$

(3). Heat equ. (1D)

$K\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ 0<x<L t>0

I.C. $u(x,0) = f(x)$ 0<x<L

B.C. $u(0,t) = 0$ $u(L,t) = 0$ t>0



u:temprature

$$K = \frac{K}{\mu\rho}$$

Sol. Using the method of Seperation of variable , we have

$u(x,t) = X_{(x)}T_{(t)}$

$\therefore KX''T = XT'$

$$\frac{x''}{x} = \frac{T'}{KT} = \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases}$$

1. $\lambda^2 > 0$ Two O.D.E.s

$$\begin{cases} x'' - \lambda^2 x = 0 \\ T' - k\lambda^2 T = 0 \end{cases}$$

Boundary conditions

$$\begin{cases} u(0,t) = x(0)T(t) = 0 \\ u(L,t) = x(L)T(t) = 0 \end{cases}$$

$$\therefore X(0) = 0, X(L) = 0$$

$$\begin{cases} x'' - \lambda^2 x = 0 \\ X(0) = 0, X(L) = 0 \end{cases} sturm - Liouville\_problem$$

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

$$\begin{cases} X(0) = C_1 + C_2 = 0 \\ X(L) = C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0 \end{cases} \Rightarrow C_1 = C_2 = 0$$

$$X(x) = 0\_ \therefore u(u,t) = X(x)T(t) = 0\_trival\_solution$$

2. $\lambda = 0$

$$\begin{cases} x'' = 0 \\ T' = 0 \end{cases} X(x) = 0, X(L) = 0$$

$$X(x) = C_1 x + C_2$$

$$\begin{cases} X(0) = C_2 = 0 \\ X(L) = C_1 L = 0 \Rightarrow C_1 = 0 \end{cases} \therefore X(x) = 0 \Rightarrow u(x,t) = 0\_trival\_solution$$

3

$$-\lambda^2 < 0$$

$$\frac{X''}{X} = \frac{T'}{KT} = -\lambda^2 \_Boundary\_condition \begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$$

$$Two\_O.D.E.s$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + K\lambda^2 T = 0 \end{cases}$$

$$X(x) = C1 \cos \lambda X + C2 \sin$$

$$\begin{cases} X(0) = C1 = 0 \\ X(L) = C2 \sin \lambda L = 0 \end{cases}$$

$$For\_nontrival\_solution\_C2 \neq 0$$

$$\therefore \sin(\lambda L) = 0$$

$$\lambda L = n\pi \_n = 1.2.3...$$

$$\therefore \lambda = \frac{n\pi}{L}, n = 1.2.3.....eigenvalue$$

$$X(x) = C2 \sin(\frac{n\pi}{L} x)\_n = 1.2.3....eigenfunction$$

$$T' = K\lambda^2 T = 0$$

$$T(t) = C3 e^{-K\lambda^2 t} = C3 e^{-K(\frac{n\pi}{L})^2 t}$$

$$\therefore u_n(x.t) = X(x)T(t) = A_n e^{-K(\frac{n\lambda}{L})^2 t} \sin(\frac{n\pi}{L}x) \_ n = 1.2.3....$$

$\therefore general\_solution(surperposition\_principle)$

$$u(x.t) = \sum_{n=1}^{\infty} u_n(x.t) = \sum_{n=1}^{\infty} A_n e^{-K(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi}{L}x)$$

$A_n = ?$

$Initial\_condition$

$$u(x.0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)$$

$Half-range.\_\exp ansion\_of\_f(x)\_in\_a\_Fourier\_\sin e\_series$

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$$

$Other\_problems$

(i)

$Keumann\_B.C.$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0.\_\frac{\partial u}{\partial x}\Big|_{x=L} = 0$$

(ii)

$Mixed\_B.C.$

$$u(0.t) = 0\_\frac{\partial u}{\partial x}\Big|_{x=L} = 0$$

(iii)

$Semi-\inf inite\_rod\_0 < x < \infty$

$B.C.\_u(0.t) = 0\_t > 0$

$I.C.\_u(x.0) = f(\bar{x})\_0 < x < \infty$

$u(x.t)\_is\_bounded\_at\_x = \infty$

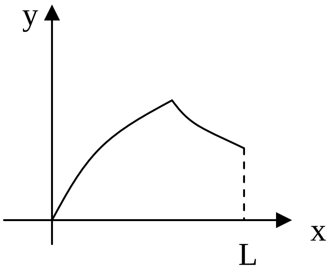
(iv)

$\inf inite\_rod\_-\infty < x < \infty$

$u(x.t)\_bounded\_at\_x = \pm\infty$

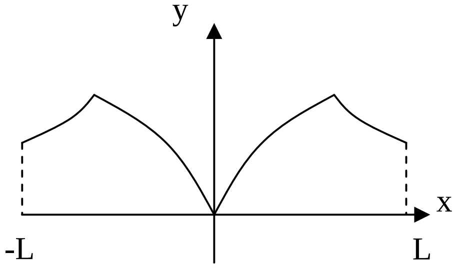
Half range expansion

$y = f(x)$ is defined on $[0, L]$



(1) cosine half range expansion

Reflect the graph $f(x)$ about y-axis onto $-L < x < 0$; then function is even on $[-L, L]$



$f(x)$ is defined $[-L, L]$ and $f(x) = f(-x)$

$\therefore$ $f(x)$ can be expaned in Fourier cosine series (cosine half-range expansion)

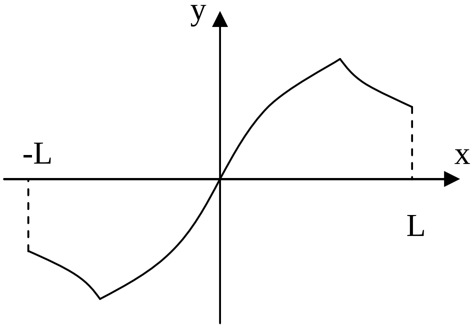
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

(2) sine half-range expansion

Reflect the graph $f(x)$ through the origin onto $-L \le x \le 0$; then new function is odd on $[-L, L]$



$f(x)$ is defined on $[-L, L]$ and $f(-x) = -f(x)$

$\therefore\ f(x)$ can be expaned in Fourier sine series naimly

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

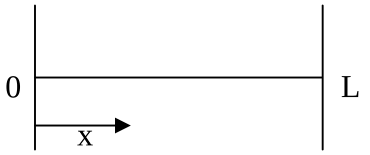
$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

(4) Wave equation

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \qquad 0<x<L\,,t>0$$

BC's $u(0,t)=0, u(L,t)=0, t>0$

IC's $u(x,0)=f(x), \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)\,, 0<x<L$



The motion of an elastic string stretched between two pegs is studied if the string is lifted and released to virbrated in a plane

$$a^2=\frac{h}{\rho}$$

h = horizontal tension of string

ρ = density

Solution:

(i) Two O.D.E.s

$$u(x,t)=X(x)T(t)$$

$$\therefore \frac{X''}{X}=\frac{T''}{a^2T}=-\lambda^2<0$$

$(a^2=0,\lambda^2>0), u(x,t)=0$

$$\begin{cases} X''+\lambda^2 X=0 \\ T''+a^2\lambda^2 T=0 \end{cases}$$

(ii) Satisfying BC's

General solution for the two O.D.E.s

$\{X(x)=c_1\cos\lambda x+c_2\sin\lambda x$

$T(t)=c_3\cos a\lambda t+c_4\sin a\lambda t$

Boundary condition

$u(0,t) = X(0)T(t) = 0$

$u(L,t) = X(L)T(t) = 0$

$\Rightarrow X(0) = 0, X(L) = 0$

$X(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$

$X(L) = c_2 \sin \lambda L = 0$

for nontrivial solution, $c_2 \neq 0$

$\therefore \sin \lambda L = 0$

$\lambda L = n\pi \quad n = 1,2,3\ldots$

eigenvalue $\lambda_n = \dfrac{n\pi}{L}, n = 1,2,3\ldots$

eigenfunction
$$X_n = \sin \frac{n\pi}{L} x, n = 1,2,3\ldots$$
$$T_n = c_3 \cos \frac{n\pi}{L} at + c_4 \sin \frac{n\pi}{L} at$$

(iii) General solution

$$u_n(x,t) = X_n T_n = (A_n \cos \frac{n\pi}{L} at + B_n \sin \frac{n\pi}{L} at) \sin \frac{n\pi}{L} x, n = 1,2,3\ldots$$

$\therefore$ general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi}{L} at + B_n \sin \frac{n\pi}{L} at) \sin \frac{n\pi}{L} x$$

$A_n = ?\ B_n = ?$

The solution must satisfy initial condition

$$u(x,o) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \qquad 0 < x < L$$

sine half-range expansion

$$\therefore A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-A_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} at + B_n \frac{n\pi a}{L} \cos \frac{n\pi}{L} at) \sin \frac{n\pi}{L} x$$

$$\therefore \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} (B_n \frac{n\pi a}{L}) \sin \frac{n\pi}{L} x$$

$$\therefore B_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx$$

(5) Laplace equation

stady-state two-dimensional heat flow

$$\frac{\partial u}{\partial t} = 0$$

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

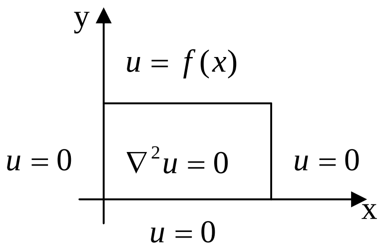
$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$0 < x < a \qquad 0 < y < b$

Boundary conditions

$u(0, y) = 0 \qquad u(a, y) = 0 \qquad 0 < y < b$

$u(x,0) = 0 \qquad u(x,b) = f(x) \qquad 0 < x < a$



Solution:

$u(x, y) = F(x)G(y) \qquad$ BC's $\quad F(0) = 0\,, F(a) = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda^2$$

$$\therefore F(x) = c_1 \cos\lambda x + c_2 \sin\lambda x$$

$$F(0) = c_1 = 0$$

$$F(a) = c_2 \sin\lambda a = 0$$

For nonzero solution $\;c_2 \neq 0$

$\therefore \sin\lambda a = 0$

eigenvalue $\quad \lambda_n = \dfrac{n\pi}{a} \qquad n = 1,2,3\cdots$

eigenfunction $\quad F_n = \sin\dfrac{n\pi}{a}x \qquad n = 1,2,3\cdots$

$G'' - \lambda^2 G = 0$

B.C. $\quad G(0) = 0$

$$G(y) = c_3 \cosh\lambda y + c_4 \sinh\lambda y$$

$G(0) = c_3 = 0 \qquad \therefore G_n(y) = c_4 \sinh\dfrac{n\pi}{a}y \qquad n = 1,2,3\cdots$

$\therefore$ eigenfunction $\;u_n(x, y)\;$ is

$u_n(x, y) = F_n(x)G_n(y)$

$$= A_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y \qquad n = 1,2,3\cdots$$

By superposition principle, the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$$

using boundary condition $u(x,b) = f(x)$, we have

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} (A_n \frac{n\pi b}{a}) \sin\frac{n\pi}{a}x \qquad 0 < x < a$$

$$\therefore b_n = A_n \sin\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}xdx$$

$$A_n = \frac{2}{a\sin\frac{n\pi b}{a}}\int_0^a f(x)\sin\frac{n\pi}{a}xdx$$

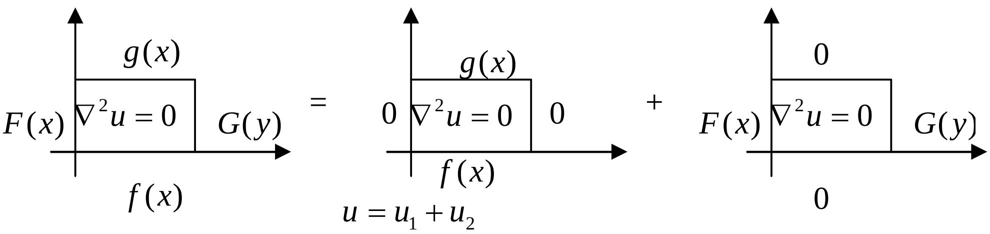
Other boundary conditions.

(i) Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

$$\begin{cases} u(0,y) = F(y), u(a,y) = G(y) \\ u(x,0) = f(x), u(x,b) = g(x) \end{cases} \qquad 0 < x < a \qquad 0 < y < b$$

Hint, superposition principle



$u = u_1 + u_2$

(ii) Mixed boundary condition

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

$$u = (x,0) = 0 \quad , u(x,b) = f(x) \qquad 0 < y < b$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0 \quad , \quad \frac{\partial u}{\partial x}\Big|_{x=a} = 0 \qquad 0 < x < a$$

Sol $\quad u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\lambda y \cos\lambda x$

$$\lambda = \frac{n\pi}{a}, \quad n = 1,2,3\cdots$$

$$\begin{cases} A_0 = \dfrac{1}{ab}\displaystyle\int_0^a f(x)dx \\ A_n = \dfrac{2}{a\sinh\lambda b}\displaystyle\int_0^a f(x)\cos\lambda x dx \end{cases}$$

(6) Nonhomogeneous equation and boundary condition

$$\begin{cases} k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t} \\ u(0,t) = k_1 \ , \ u(L,t) = k_2 \end{cases}$$

P.D.E is not separable!!

method: A change of dependent variable

$u(x,t) = \nu(x,t) + \psi(x)$

Where $L\nu = 0$ with homogeneous B.C's and ψ (x) is to be determined

EX: $\kappa\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$ 0<x<1, t>0

B.C's: $u(0,t) = 0 \ , \ u(1,t) = u_0 \quad t > 0$

I.C's: $u(x,0) = f(x)$

Sol: Let $u(x,t) = \nu(x,t) + \psi(x)$

$$\therefore \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 \nu}{\partial x^2} + \psi''(x) \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial \nu}{\partial t} \end{cases}$$

Substituting above equations into the P.D.E, we have

$$\kappa\frac{\partial^2 \nu}{\partial x^2} + \kappa\psi'' + r = \frac{\partial \nu}{\partial t}$$

Assume

$$\kappa\frac{\partial^2 \nu}{\partial x^2} = \frac{\partial \nu}{\partial t}$$

And

$\nu(0,t) = 0 \ , \ \nu(1,t) = 0$

$\therefore \kappa\psi'' + r = 0$

$$=> \psi''(x) = -\frac{r}{2k}x^2 + c_1 x + c_2$$

Furthermore

$u(0,t) = \nu(0,t) + \psi(0) = 0$

$u(1,t) = \nu(1,0) + \psi(1) = u_0$

$$\therefore \begin{cases} \psi(0) = 0 \\ \psi(1) = u_0 \end{cases}$$

$\therefore \psi(0) = c_2 = 0$

$$\psi(1) = -\frac{r}{2k}x^2 + c_1 = u_0$$

$$c_1 = u_0 + \frac{r}{2k}$$

$\therefore \psi(x) = -\frac{r}{2k}x^2 + (u_0 + \frac{r}{2k})x$

Initial condition

$u(x,0) = v(x,0) + \psi(x) = f(x)$

$\therefore v(x) = f(x) - \psi(x)$

$= f(x) + \frac{r}{2k}x^2 - (u_0 + \frac{r}{2k})x$

$\therefore v(x,t)$ satisfies the following homogeneous equation and condition.

$$\begin{cases} \kappa \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad 0 < x < 1 \ , \ t > 0 \\ \text{B.C's} : v(0,t) = 0 \ , \ v(1,t) = 0 \quad t > 0 \\ \text{I.C's} : v(x,0) = f(x) - \psi(x) \end{cases}$$

$\therefore v(x,t)$ can be solved by using the method of separation of variable

Thus

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{\kappa n^2 \pi^2 t} \sin(n\pi x)$$

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin(n\pi x)dx$$

$\therefore u(x,t) = v(x,t) + \psi(x)$

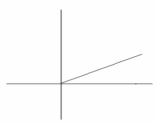
$$= \sum_{n=1}^{\infty} A_n e^{\kappa n^2 \pi^2 t} \sin(n\pi x) + \frac{r}{2k}x^2 + (\frac{r}{2k} + u_0)x$$

3. Boundary-value Problems in other Coordinate System

temperature in	circular disk.	$\rightarrow$	polar coordinate
	circular cylinder	$\rightarrow$	cylindrical coordinate
	sphere	$\rightarrow$	spherical coordinate

(1) Problems involving Laplace equation in Polor coordinates

Laplace in polor coordinate



$x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$

2D Laplace

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \sin\theta \frac{\partial u}{\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r^2}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r^2}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1. B.V.P in other Coordinate System
   (1) Laplacian in Polar Coordinate
   (2) Circular Membrane : Fourier – Bessel Series
      drums, pump, microphone, telephone……..

   2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = C^2 (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

   In polar coordinate $u(r,\theta,t)$

$$\frac{\partial^2 u}{\partial t^2} = C^2 (\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2})$$

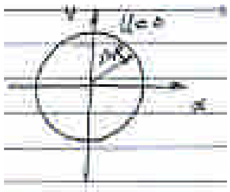
   Consider the radial symmetry problem, namely, $\theta - independent$

$$\frac{\partial^2 u}{\partial t^2} = C^2 (\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) \qquad 0 < r < R, \quad t > 0$$

B.C $u(R,t) = 0 \quad t > 0$

I.C $u(r,0) = f(r)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r) \qquad 0 < r < R$$



Sol:

(i) O.D.E. Bessel's Equation

$$u(r,t) = W(r)G(t)$$

$$\frac{\ddot{G}}{C^2 G} = \frac{1}{W}(W'' + \frac{1}{r}W') = -k^2$$

$\therefore$ 2 O.D.Es

$$\ddot{G} + \lambda^2 G = 0 \qquad \lambda = \text{ck}$$

$$W'' + \frac{1}{r}W' + k^2 W = 0$$

or $r^2 W'' + rW' + k^2\lambda^2 W = 0$

A parametric Bessel D.E with parametric $\upsilon = 0$

(ii) Satisfying BCs

General solutions

$W(r) = C_1 J_0(kr) + C_2 Y_0(kr)$ (See handout)

$G(t) = C_3 \cos \lambda t + C_4 \sin \lambda t$

Recall that

$\lim_{r \to 0^+} Y_0(kr) = -\infty$ and $W(\theta)$ is bounded,

$\therefore C_2 = 0$

$u(R,t) = 0 \Rightarrow W(R) = 0$

$\therefore C_1 J_0(KR) = 0 \Rightarrow J_0(KR) = 0 \ (C_1 \neq 0)$

$K_m R = \alpha_m \qquad m = 1,2,3,....... \qquad K_m = \alpha_m / R$

$\alpha_1 = 2.4048$, $\alpha_2 = 5.5201$, $\alpha_3 = 8.6537$, $\alpha_4 = 11.7915$,.......

Eigenfuction and Eignvalue

$$W_m(r) = C_1 J_0(K_m r) = C_1 J_0(\frac{\alpha_m}{R} r)$$
$$G_m(t) = C_3 \cos\lambda_m t + C_4 \sin\lambda_m t \qquad (\lambda_m = cK_m)$$
$$\therefore U_m(r,t) = W_m(r) G_m(t)$$
$$= (a_m \cos\lambda_m t + b_m \sin\lambda_m t) J_0(K_m r)$$

(iii) Final solution

Superposition principle

$$u(r,t) = \sum_{m=1}^{\infty} (a_m \cos\lambda_m t + b_m \sin\lambda_m t) J_0(\frac{\alpha_m}{R} r)$$
$$u(r,\theta) = f(r) = \sum_{m=1}^{\infty} a_m J_0(\frac{\alpha_m}{R} r)$$

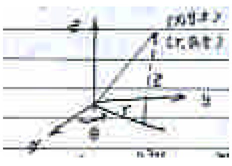
Recall the Fourier - Bessel series (see handout)

If $J_0(K_m R) = 0$

$$a_m = \frac{2}{R^2 J_1{}^2(k_m R)} \int_0^R r f(r) J_0(k_m r) dr$$
$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} (-\lambda_m a_m \sin\lambda_m t + \lambda_m b_m \cos\lambda_m t) J_0(k_m r)$$
$$U_t(r,0) = g(r) = \sum_{m=1}^{\infty} \lambda_m b_m J_0(k_m r)$$
$$\therefore b_m = \frac{2}{\lambda_m R^2 J_1{}^2(k_m R)} \int_0^R r g(r) J_0(k_m r) dr$$

(2) Laplacian in cylindrical and Spherical Coordinate

Cylindrical coordinate $R\ \theta\ Z$



$$X = r\cos\theta$$
$$Y = r\sin\theta$$
$$Z = Z$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial Z^2}$$

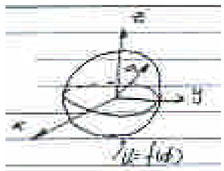
Spherical Coordinate $R\ \theta\ \phi$

$$X = r\cos\theta\sin\phi$$
$$Y = r\sin\theta\sin\phi$$
$$Z = Z\cos\phi$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \phi} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

Steady-state temperature of sphere: Legendre polynomial

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi} = 0 \qquad \begin{array}{l} 0 < r < a \\ 0 < \phi < \pi \end{array}$$



$$u(a,\phi) = f(\phi) \qquad 0 < \phi < \pi$$

u is independent of θ

$u(0,\phi)$ is bounded

Sol.

$$u(r,\phi) = G(r)H(\phi)$$
$$\frac{1}{G}(r^2G'' + 2rG') = -\frac{1}{H}(H'' + \cot\phi H') = \lambda^2$$
$$r^2G'' + 2rG' - \lambda^2 G = 0 \qquad \text{Euler - Candy D.E.} \quad (\lambda^2 = n(n+1))$$
$$\therefore G(r) = C_1 r^n + C_2 r^{-(n+1)} \qquad \text{(see sec 2.6)}$$

$$\sin\phi H'' + \cos\phi H' + \lambda^2 \sin\phi H = 0$$

Let $w = \cos\phi$, $0 \le \phi \le \pi$ we have

$$(1\text{-}w^2)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + n(n+1)H = 0 \quad (\lambda^2 = n(n+1))$$
$$-1 \le w \le 1$$

Legendre's Equation

Because $u(r,\phi)$ bounded at $r = 0$ $\Rightarrow C_2(0) = 0$ is bounded

$$\Rightarrow C_2 = 0$$

$$\therefore \quad G_n = C_1 r^n$$
$$\therefore \quad U_n(r,\phi) = G_n(r)H_n(\phi) = A_n r^n p_n(\cos\phi)$$
$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n p_n(\cos\phi)$$

$$u(\theta,\phi) = f(\phi) = \sum_{n=0}^{\infty} A_n a^n p_n(\cos\phi) \qquad \text{Fourier - Legendre Series}$$

$$A_n = \frac{2n+1}{2a^n}\int_0^{\pi} f(\phi)\, p_n(\cos\phi)\sin\phi\, d\phi \qquad \text{(see handout)}$$

$$u(r,\phi) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi} f(\phi)\, p_n(\cos\phi)\sin d\phi\right)\left(\frac{n}{a}\right)^n p_n(\cos\phi)$$

Conclusions :

P.D.E in other Coordinate

Polar Coordinate
Cylindrical Coordinate
Spherical Coordinate

Separation of variables

O.D.E with variable coefficients

Canchy-Eula D.E
Bessel D.E
Legrendre D.E

Fourier-Bessel series
Fourier-Legrendre series