Ⅱ.Sturm-Liouville problem

Two point boundary- value problem

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0 \qquad a\le x\le b$$

p(x) , q(x) , r(x) , $r'(x)$ are continuous and p(x)>0 on $a\le x\le b$

Boundary conditions

$$\begin{cases}\alpha_1 y(a)+\beta_1 y'(a)=0\\ \alpha_2 y(b)+\beta_2 y'(b)=0\end{cases}$$

$\alpha_1,\beta_1,\alpha_2,\beta_2$ are constants

The nontrivial solution y(x) of above problem is called eigenfunction and λ is called eigenvalue corresponding to eigenfunction y(x).

EX: If r(x)=1 , g(x)=0 , p(x)=1

then

$$y''+\lambda y=0$$

1. Regular sturm-liouville problem

   Condition: r(x)>0, p(x)>0, on [a , b] let $y_m(m)$ and $y_n(x)$ be eigenfunction corresponding to eigenvalue λ_m and λ_n.

   Then ① $\int_a^b p(x)y_m(x)y_n(x)dx=0 \qquad \lambda_m+\lambda_n$ ....orthogronality of y_m & y_n

   ② $\|y_m\|=\sqrt{\int_a^b p(x)y_m^2(x)dx}$ .... Norm of $y_m(x)$

   Proof of orthogonality of eigenfunctions

   $\because y_m(x)$, $y_n(x)$ are solutions of sturm-Liouville eqn.

   $$\therefore \frac{d}{dx}[r(x)y_m(x)]+[q(x)+p(x)]y_m(x)=0 \;.....①$$

   $$\frac{d}{dx}[r(x)y_n(x)]+[q(x)+p(x)]y_n(x)=0 \;....②$$

   $①\times y_n - ②\times y_m$ gives

   $$(\lambda_m-\lambda_n)p(x)y_m(x)y_n(x)=y_m\frac{d}{dx}[r(x)y_n']-y_n\frac{d}{dx}[r(x)y_m']$$

   Integrate both sides w.r.t. x form a to b to have

$$(\lambda_m-\lambda_n)\int_a^b p(x)y_m y_n dx=\int_a^b y_m \frac{d}{dx}[r(x)y_n']dx-\int_a^b y_n \frac{d}{dx}[r(x)y_m']dx$$

分部積分 $$=r(x)y_m y_n'\Big|_a^b-\int_a^b r(x)y_n'y_m'dx-r(x)y_n y_m'\Big|_a^b+\int_a^b r(x)y_m'y_n'dx$$

$$=r(b)y_m(b)y_n'(b)-r(a)y_m(a)y_n'(a)-r(b)y_n(b)y_m'(b)+r(a)y_n(a)y_m'(a)$$
$$=r(b)[y_m(b)y_n'(b)-y_n(b)y_m'(b)]-r(a)[y_m(a)y_n'(a)-y_n(a)y_m'(a)]$$

From vounding condition, we have $\begin{cases}\alpha_1 y_m(a)+\beta_1 y_m'(a)=0\\ \alpha_1 y_n(a)+\beta_1 y_n'(a)=0\end{cases}$

$\because$ α_1,β_1 are not all zeros

$\therefore$ $\begin{vmatrix} y_m(a) & y_m'(a)\\ y_n(a) & y_n'(a)\end{vmatrix}=0$ => $y_m(a)y_n'(a)-y_n(a)y_m'(a)=0$

Similarly :

$$y_m(b)y_n'(b)-y_n(b)y_m'(b)=0$$

and $r(a)\neq 0, r(b)\neq 0$

$\therefore$ $(\lambda_m-\lambda_n)\int_a^b p(x)y_m(x)y_n(x)dx=0$

if $\lambda_m\neq\lambda_n$

then $\int_a^b p(x)y_m(x)y_n(x)dx=0$

$\therefore$ $y_m(x)$ & $y_n(x)$ are orthogonal w.r.t. weight fn. $p(x)$

2. Singular sturm-Liouville

(1) $r(a)=0, r(b)\neq 0$

$\alpha_1 y(a)+\beta_1 y'(a)=0$ can be dropped

$\alpha_2 y(b)+\beta_2 y'(b)=0$ retained

(2) $r(a)\neq 0, r(b)=0$

$\alpha_2 y(b)+\beta_2 y'(b)=0$ can be dropped

$\alpha_1 y(a)+\beta_1 y'(a)=0$ retained

(3) $r(a)=r(b)=0$

No boundary condition are specified, but solutions must be bounded on $[a,b]$

If $r(a)=r(b)\neq 0$ then periodic boundary condition $y(a)=y(b)$, $y'(a)=y'(b)$ are used.

3. Self - adjoint Form

$$a(x)y''+b(x)y'+[c(x)+\lambda d(x)]y=0$$

$a(x)\neq 0$, and $a(x), b(x)$, $c(x)$, $d(x)$ are continuous on $[a,b]$

Multiplied above D.E. by an integration factor $\frac{1}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}$, the D.E. can be expressed in the form of Storm-Liouville equation.

Ex.

1. Bessel D.E.

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0$$

Self-adjoint form?

Integration factor

$$u(x) = \frac{1}{x^2}e^{\int \frac{x}{x^2}dx} = \frac{1}{x^2}e^{\ln x} = \frac{1}{x}$$

$$\frac{1}{x}[x^2y'' + xy'] + \frac{1}{x}(k^2x^2 - n^2)y = 0$$

$$\frac{d}{dx}[xy'] + \left(-\frac{n^2}{x} + \lambda x\right)y = 0$$

$r(x) = x$

$q(x) = -\frac{n^2}{x}$

$p(x) = x$

$\lambda = k^2$

2. Legendre D.E.

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

Self-adjoint form?

Integration factor

$$u(x) = \frac{1}{1-x^2}e^{\int \frac{-2x}{1-x^2}dx} = \frac{1}{1-x^2}e^{\ln|1-x^2|} = 1$$

$$\therefore \frac{d}{dx}[(1-x^2)y] + \lambda y = 0$$

$r(x) = 1 - x^2$

$q(x) = 0$

$p(x) = 1$

$\lambda = n(n+1)$

4. Solutions of Sturm-Liouville equation

D.E.

$$y' + \lambda y'' = 0 \qquad (r(x) = 1, \; q(x) = 0, \; p(x) = 1) \qquad -\pi < x < \pi$$

Boundary conditions

$$y(\pi) = y(-\pi)$$
$$y'(\pi) = y'(-\pi) \qquad \text{Periodic BCs}$$
$$-\pi \le x \le \pi$$

sol /

characteristic equation

$S^2+\lambda=0$

① if $\lambda=-k^2$

$\therefore S^2-k^2=0$

$S=\pm k$ 相異實根

$\therefore y_{(x)}=A\cosh(kx)+B\sinh(kx)$

$y'_{(x)}=Ak\sinh(kx)+B\sinh(kx)$

$y_{(x)}=A\cosh(kx)+B\sinh(kx)$

$=y_{(\pi)}$

$=A\cosh(-kx)+B\sinh(-kx)$

$=A\cosh(kx)-\sinh(kx)$

$\therefore 2B\sinh(k\pi)=0$

$\therefore B=0 \quad (\sinh(k\pi)\neq 0)$

From $y'_{(\pi)}=y'_{(-\pi)}$,

we have A=0

$\therefore y_{(x)}=0$ trivial solution

② if $\lambda=0$

$S=0 \quad y''=0$

$y_{(x)}=Ax+B$

$y_{(\pi)}=A\pi+B=y_{(-\pi)}=-A\pi+B$

$\therefore A=0$

$y'_{(x)}=B=y'_{(\pi)}=y'_{(-\pi)}$

$\therefore$solution $y_{(x)}=B$

③ $\lambda=k^2>0$

$S^2+k^2=0$

$S=\pm \iota k$ 共軛虛根

$y_{(x)}=A\cosh(kx)+B\sinh(kx)$

$y'_{(x)}=-A\cosh(kx)+B\sinh(kx)$

Boundary conditions

$y(\pi)=y(-\pi)$

Acos(kπ)+Bsin(kπ)= Acos(-kπ)+Bsin(-kπ)

= Acos(kπ)-Bin(kπ)

$\therefore$ 2 Bsin(kπ)=0

y'(π)=y'(-π)

Aksin(kπ)+Bkcos(kπ)= Aksin(kπ)+Bkcos(kπ)

$\therefore$ 2Aksin(kπ)=0

$\therefore$ 2 Bsin(kπ)=0

2Aksin(kπ)=0

For nontrivial solutions A$\neq$0 B$\neq$0

$\therefore$ sin(kπ)=0

$\therefore$ kπ=nπ ， n=1，2，3●●●

$\therefore$ $\lambda = k^2$ ●●● eigenvalue k=1，2，3●●●

$\therefore$ the eigenfunction y(x)

corresponding to the eigenvalue $\lambda = k^2$

i.e y(x)=Acos$\sqrt{\lambda}$ x+Bsin$\sqrt{\lambda}$ x

$\therefore$

eigenvalue	0	1	4	9
eigenfuntion	1	cosx,sinx	Cos2x,sin2x	Cos3x,sin3x

The set of eigenfunction {1 , cosmx , sinmx }

m=1 , 2 , 3 , ……on [-π , π] is orthogonal (P| x | = 1)

Othogonality of 1 , cosmx & sinmx

$$(1, \cos mx) = \int_{-\pi}^{\pi} \cos mx dx = 0$$

$$(1, \sin mx) = \int_{-\pi}^{\pi} \sin mx dx = 0$$

$$(\cos mx, \sin nx) = \int_{-\pi}^{\pi} \cos mx \sin nx dx = 0(m \neq n)$$

$$(\cos mx, \cos nx) = \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0(m \neq n)$$

$$(\sin mx, \sin nx) = \int_{-\pi}^{\pi} \sin mx \sin nx dx = 0(m \neq n)$$

Norms of 1 , cosmx   sinmx

$$\|1\| = \sqrt{\int_{-\pi}^{\pi} 1^2 dx} = \sqrt{2\pi}$$

$$\|\cos mx\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 mx dx} = \sqrt{\pi}$$

$$\|\sin mx\| = \sqrt{\int_{-\pi}^{\pi} \sin^2 mx dx} = \sqrt{\pi}$$

Hense , an orthogonal set of eigenfunction { 1 , cosmx , sinmx }
is $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos mx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}} \right\}$

5. eigenfunction Expansion

Can we expand a function f(x) into a series of orthogonal function { Φn(x) }?

$$f(x) = C_0 \Phi_0(x) + C_1 \Phi_1(x) + ...C_n \Phi_n(x) + ...$$

$$= \sum_{n=1}^{\infty} C_n \Phi_n(x) \quad \text{on [a , b]}$$

Thus is call an orthorgonal expansion or generalized Fouries series

If $\Phi_n(x)$ are eigenfunctions of a Sturm – Liouville problem , it is

Called an eigenfunction expansion

[proof] multiphy eqn.1 by $\Phi_n(x)$ and integrate over [a , b] to give

$$\int_a^b f(x)\Phi_n(x)dx = \sum_{n=0}^{\infty} C_n \int_a^b \Phi_n(x)\Phi_m(x)dx = 0(m \neq n)$$

$$= C_m \int_a^b \Phi_m^2(x)dx \quad (\text{m} \neq \text{n})$$

$$= C_m \|\Phi_m\|^2$$

$$\therefore \qquad C_m = \frac{\int_a^b f(x)\Phi_m^2(x)dx}{\|\Phi_m(x)\|^2} \quad \text{.......Fourier constant}$$

$$f(x)=\sum_{n=1}^{\infty}\frac{\int_a^b \Phi_m^2(x)dx}{\left\|\Phi_m(x)\right\|^2}\Phi_n(x)$$

If { Φn(x) } is orthogonal w.r.t p(x) on [a , b]

$$C_n=\frac{\int_a^b p(x)f(x)\Phi_m^2(x)dx}{\left\|\Phi_m(x)\right\|^2}$$

and $\left\|\Phi_n\right\|^2=\int_a^b p(x)\Phi_n^2(x)dx$

Ex Sturm – Liouville problem

$y^{//}+\lambda y=0, y(\pi)=y(-\pi), y^{/}(\pi)=y^{/}(\pi)$

gives an orthogonal set { 1 , cosmx , sinmx }
m= 1 , 2 , 3 ……. On the interval $-\pi \leqq x \leqq \pi$ with p(x) = 1

Hence , a corresponding eigenfunction expansion for any function f(x) can be written as

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos mx+b_n\sin mx$........ Fourier series of f(x)

$$\begin{cases} a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx \\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx \\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx \end{cases}$$

H.W.
Given : a sturm – liouville problem

$$\begin{cases} y^{//}+\lambda y=0 \\ y(0)=0, y(L)=0 \end{cases}$$

Find : 1. solve the D.E and Find its eigenfunctions & expansion
2.prove the orthogonality of eigenfuntion are evaluate its norm
3.write down the corresponding eigenfunction expansion for any function f(x)

§ Bessel's Equation and Bessel's Function

1. Bessel's differential equation

$$x^2 y'' + (x^2 - \upsilon^2)\, y = 0 \qquad \upsilon \geq 0$$

or

$$y'' + \left(1 - \frac{\upsilon^2}{x^2}\right) y = 0$$

the general solution on $0 < x < \infty$ is

$$y(x) = C_1 J_\upsilon(x) + C_2 J_{-\upsilon}(x) \qquad \upsilon \neq \text{integer}$$

where J_υ and $J_{-\upsilon}$ are Bessel's function of the first kind

$$J_\upsilon(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$J_{-\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n-\upsilon}$$

Here $\Gamma(n)$ is Gamma function defined as

$$\Gamma(n) = \int_0^{\infty} u^{n-1} e^{u}\, du \quad , \quad n > 0$$

and $\Gamma(n+1) = n\Gamma(n)$

EX.

$$x^2 y'' + x y' + \left(x^2 - \tfrac{1}{4}\right) y = 0$$

$$\upsilon^2 = 1/4 \quad , \quad \upsilon = 1/2$$

$$\therefore\ y(x) = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x)$$

Bessel Function of the second kind

Define.

$$Y_\upsilon(x) = \frac{\cos\upsilon J_\upsilon(x) - J_{-\upsilon}(x)}{\sin\upsilon\pi} \qquad \text{(Neumann's function)}$$

If $\upsilon \neq$ integer

$Y_\upsilon(x)$ and $J_\upsilon(x)$ are linear independent

If $\upsilon \to \mathrm{m} = \text{integer}$

$$\mathrm{Y}_{\upsilon}(x) = \lim_{\upsilon \to m} \mathrm{Y}_{\upsilon}(x) \qquad (\text{ L`Hopital's rule })$$

Y_m and J_m are linearly independent

Hence, the general solution of Bessel's D.E for any υ is expressed as

$$y(x) = \mathrm{C}_1 \mathrm{J}_{\upsilon}(\mathrm{x}) + \mathrm{C}_2 \mathrm{Y}_{\upsilon}(\mathrm{x})$$

Ex.

$$\mathrm{x}^2 y'' + \mathrm{x} y' + (\mathrm{x}^2 - 9)\, y = 0$$

$$\upsilon^2 = 9 \quad , \quad \upsilon = 3$$

$$\therefore\ y(x) = \mathrm{C}_1 \mathrm{J}_3(\mathrm{x}) + \mathrm{C}_2 \mathrm{Y}_3(\mathrm{x})$$

2. Parametric Bessel Differential Equation

$$\mathrm{x}^2 y'' + \mathrm{x} y' + (\lambda^2 \mathrm{x}^2 - \upsilon^2)\, y = 0 \qquad \text{let } \lambda \mathrm{x} = \mathrm{s}$$

general solution

$$y(x) = \mathrm{C}_1 \mathrm{J}_{\upsilon}(\lambda \mathrm{x}) + \mathrm{C}_2 \mathrm{Y}_{\upsilon}(\lambda \mathrm{x})$$

3. Properties of Bessel function of order m

(1) $\mathrm{J}_{\text{-m}}(x) = (-1)^m \mathrm{J}_m(x)$

(2) $\mathrm{J}_{\mathrm{m}}(-x) = (-1)^m \mathrm{J}_m(x)$

(3) $\mathrm{J}_{\mathrm{m}}(0) = 0 \qquad m > 0$

(4) $\mathrm{J}_0(0) = 1$

(5) $\lim_{\mathrm{x} \to 0^+} \mathrm{Y}_{\mathrm{m}}(x) = -\infty$

4. Bessel function of the third kind

$$\mathrm{H}_{\upsilon}^{(1)}(\mathrm{x}) = \mathrm{J}_{\upsilon}(\mathrm{x}) + i\mathrm{Y}_{\upsilon}(\mathrm{x})$$

$$\mathrm{H}_{\upsilon}^{(2)}(\mathrm{x}) = \mathrm{J}_{\upsilon}(\mathrm{x}) - i\mathrm{Y}_{\upsilon}(\mathrm{x})$$

Hankel function of the first and second kinds (wave propagation problems)

modified Bessel function

D.E

$$\mathrm{x}^2 y'' + \mathrm{x} y' - (\mathrm{x}^2 + \upsilon^2)\, y = 0$$

let $i\mathrm{x} = \mathrm{t}$, then

$$\frac{t^2 d^2 y}{dt^2} + t\frac{dy}{dt} + (t^2 - \upsilon^2) y = 0$$

(1) if υ = positive ineter and $\neq 0$

$$y(x) = C_1 J_\upsilon(ix) + C_2 J_{-\upsilon}(ix)$$

(2) otherwise υ = 0 or positive ineter

$$y(x) = C_1 J_\upsilon(ix) + C_2 Y_\upsilon(ix)$$

Modified Bessel function of the first kind

$$I_v(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\upsilon}}{k!(k+\upsilon)!} \qquad \text{real function for real } \upsilon$$

$$= i^{-\upsilon} J_\upsilon(ix)$$

Therefore

$$y(x) = C_1 J_\upsilon(ix) + C_2 J_{-\upsilon}(ix)$$

$$= C_1 I_\upsilon(ix) + C_2 I_{-\upsilon}(ix) \qquad \upsilon = \text{nonnegative integer}$$

Modified Bessel function of the 2nd kind

$$K_\upsilon(x) = \frac{\pi}{2} i^{\upsilon+1} [J_\upsilon(ik) + iY(ix) \qquad \upsilon = 0 \text{ or positive integer}$$

Then

$$y(x) = C_1 I_\upsilon(ix) + C_2 K_\upsilon(ix)$$

If $\upsilon \neq 0$ or a positive integer

$$K_\upsilon(x) = \frac{\pi}{2} \frac{I_{-\upsilon}(x) - I_\upsilon(x)}{\sin \upsilon\pi}$$

6.

(1) Differential

(i) $\frac{d}{dx}[x^{-\upsilon} J_\upsilon(x)] = -x^{-\upsilon} J_{\upsilon+1}(x)$

(ii) $\frac{d}{dx}[x^{\upsilon} J_\upsilon(x)] = x^{\upsilon} J_{\upsilon-1}(x)$

(iii) $J_{\upsilon-1}(x) - J_{\upsilon+1}(x) = 2J_\upsilon(x)$

(iv) $J_{\upsilon-1}(x) + J_{\upsilon+1}(x) = \frac{2\upsilon}{x} J_\upsilon(x)$

(2) Integral (n = integer)

(i) $\int X^n J_{n-1}(x)dx = X^n J_n$

(ii) $\int X^{-n} J_{n+1}(x)dx = -X^{-n} J_n$

§ Bessel and Legendre Series

1. Fourier-Bessel Series

Bessel D.E

$$x^2 y'' + xy' + (\lambda^2 x^2 - n^2)y = 0 \quad , n = 0,1,2,3,......$$

general solution

$$y(x) = \mathrm{C}_1 \mathrm{J}_n(\lambda \mathrm{x}) + \mathrm{C}_2 \mathrm{Y}_n(\lambda \mathrm{x})$$

self - adjoint form (Sturm - Liouville form)

$$\frac{d}{dx}[xy'] + (-\frac{n^2}{x} + \lambda^2 x)y = 0$$

$$r(x) = x,\ q(x) = -n^2/x,\ p(x) = x$$

$\because r(0) = 0$, and $\mathrm{J_n}(\lambda x)$ is bounded at x = 0

Orthogonality of Bessel function $J_n(\lambda x)$ on $0 \le x \le R$

$$\int_0^R x J_n(\lambda_i x) J_n(\lambda_i x) dx = 0 \qquad \lambda_i \ne \lambda_j$$

where eigenvalue λ are defined by a boundary condition

$$\alpha_2 J_n(\lambda R) + \beta_2 \lambda J_n'(\lambda R) = 0$$

$\{J_n(\lambda_i x)\}$ form an orthogonal net with norm $\| \mathrm{J_n}(\lambda_i x) \|$

$$\|J_n(\lambda_i x)\| = \sqrt{\int_0^R x J_n{}^2(\lambda_i x) dx}$$

The Fourier – Bessel series f(x) on (0 , R)

$$f(x) = \sum_{i=1}^{\infty} C_i J_n(\lambda_i x)$$

(1) If $J_\mathrm{n}(\lambda R) = 0$ $\quad C_i = \dfrac{2}{R^2 J_{n+2}{}^2(\lambda_i R)} \int_0^R x f(x) J_n(\lambda_i x) dx$

(2) If $hJ_\mathrm{n}(\lambda R) + \lambda R J_n'(\lambda R) = 0$ $\quad C_i = \dfrac{2\lambda_i{}^2}{(\lambda_i{}^2 R^2 - n^2 + h^2) J_n{}^2(\lambda_i R)} \int_0^R x f(x) J_n(\lambda_i x) dx$

$h \ge 0$

(3) If $J_0'(\lambda R) = 0$ $\quad f(x) = C_1 + \sum_{i=2}^{\infty} C_i J_0(\lambda_i x)$

$$C_1 = \frac{2}{R^2} \int_0^R x f(x) dx \qquad C_i = \frac{2}{R^2 J_0{}^2(\lambda_i R)} \int_0^R x f(x) J_n(\lambda_i x) dx$$

2. Fourier – Legendre Series

Legendre D.E

$$(1-x^2)y''-2xy'+n(n+1)y=0 \qquad -1\le x\le 1$$

Self – adjoint form

$$\frac{d}{dx}[(1-x^2)y']+n(n+1)y=0$$

$$r(x)=1-x^2 \ , q(x)=0 \ , p(x)=1 \ , \lambda=n(n+1), \ r(-1)=r(1)=0$$

Legendre polynomial $P_n(x)$ is eignfunction of sturm - Liouville problem of above D.E for various values of n.

Orthogonality of $P_n(x)$

$$\int_{-1}^{1} P_m(x)P_n(x)dx=0 \qquad m\ne n$$

Fourier- Legendre series of f(x) on [-1,1]

$$f(x)=\sum_{n=0}^{\infty} C_n P_n(x)$$

$$C_n=\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

Alternative Form

$$x=\cos\theta \ , dx=-\sin d\theta$$

$$F(\theta)=\sum_{n=0}^{\infty} C_n P_n(\cos\theta)$$

$$C_n=\frac{2n+1}{2}\int_{0}^{\pi} F(\theta)P_n(\cos\theta)\sin\theta d\theta$$