Chap1 Fourier series and Integrals

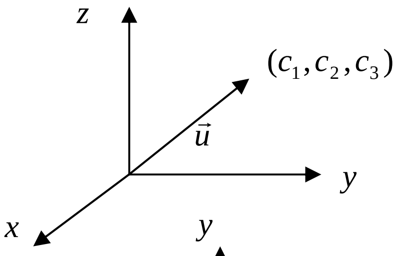
I. Orthogonal functions

(1) vectors and functions

(i) 3-D vector

$\vec{u} = (c_1, c_2, c_3)$

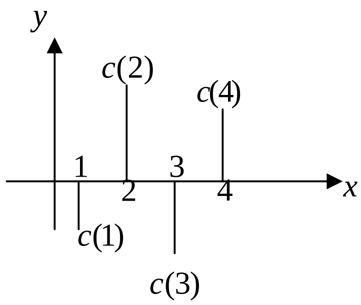
$= c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$



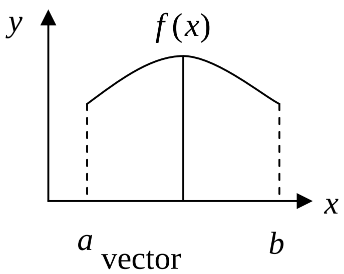
(ii) n-dimensional vector ($n > 3$)

vector

countable infinite dimensional vector



(iii) function



vector

noncountable infinite dimensional vector

A function is considered to be a generalization of a vector

(2) Orthogonality (正交性)

(i) Orthogonality of vector

(a) Inner (dot) product of vectors

$\vec{u} \cdot \vec{v} = (\vec{u}, \vec{v}) = u_1 v_1 + u_2 v_2 + \cdots$

(b) Orthogonal vector

$\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow \vec{u} \cdot \vec{v} = 0$

(ii) Othogorality of functions

(a) Inner products of $f_1(x)$ and $f_2(x)$ on $[a, b]$

$$(f_1(x), f_2(x)) = \int_a^b f_1(x) f_2(x) dx$$

(b) $f_1(x)$ and $f_2(x)$ are orthogonal on $[a, b]$

$$\xleftrightarrow{iff} (f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0$$

(c) Orthogonal set

$\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ are orthogonal set on $[a, b]$

$$\xleftrightarrow{iff} (\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x) dx = 0 \qquad m \neq n$$

(d) Orthogonal set w.r.t weight function $w(x)$

$$(\phi_m, \phi_n) = \int_a^b w(x)\phi_m(x)\phi_n(x) dx = 0 \qquad m \neq n$$

(3) Orthonormality (正規性)

(i) Norm or length of a vector

$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$

(ii) Norm of function $\phi_n(x)$ on $[a,b]$

$\|\phi_n(x)\| = \sqrt{(\phi_n,\phi_n)} = \sqrt{\int_a^b \phi_n^2(x)dx}$

(iii) Orthonormal set $\{\phi_n(x)\}$ is an orthonormal set on $[a,b]$

$\xleftrightarrow{iff}$ $\{\phi_n(x)\}$ is orthogonal set on $[a,b]$

And $\|\phi_n(x)\| = 1$ for $n = 1,2,3,\cdots$

$$\xleftrightarrow{iff} \int_a^b \phi_m(x)\phi_n(x)dx = \begin{cases} 0 \leftrightarrow m \neq n \\ 1 \leftrightarrow m = n \end{cases}$$

例題 1: show $\{1,\cos x,\cos 2x,\cdots\}$ is orthogonal on $[-\pi,\pi]$

sol: $\{\cos mx\}$ $m = 0,1,2,\cdots$ $m \neq n$

$$(\phi_m,\phi_n) = (\cos mx,\cos nx) = \int_{\pi}^{-\pi} \cos mx \cos nx dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x + \cos(m-n)x]dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)}{m-n}\right]_{-\pi}^{\pi} = 0$$ (m,n 整數 m≠n)

$\Rightarrow \{1,\cos x,\cos 2x,\cdots\}$ is orthogonal set on $[\pi,-\pi]$

例題 2: Find the norm of $\{1,\cos x,\cos 2x,\cdots\}$ on $[-\pi,\pi]$

sol: $\{\cos nx\}$ $n = 0,1,2,\cdots$

當 $n = 0$

$$\|\phi_n(x)\| = \|1\| = \sqrt{\int_{-\pi}^{\pi} 1^2 x = dx} = \sqrt{2\pi}$$

當 $n > 0$

$$\|\phi_n(x)\| = \|\cos nx\| = \sqrt{\int_{-\pi}^{\pi}\cos^2 xdx} = \sqrt{\int_{-\pi}^{\pi}\frac{1+\cos 2nx}{2}dx} = \sqrt{\pi}$$

$\therefore \left\{\frac{1}{\sqrt{2\pi}},\frac{\cos x}{\sqrt{\pi}},\frac{\cos 2x}{\sqrt{\pi}},\cdots\right\}$ is an orthonormal set on $[-\pi,\pi]$

check !

$n = 0$

$$\|\phi_0(x)\| = \left\|\frac{1}{\sqrt{2\pi}}\right\| = \sqrt{\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2\pi}}\right)^2 dx} = 1$$

$n > 0$

$$\|\phi_n(x)\| = \left\|\frac{\cos nx}{\sqrt{\pi}}\right\| = \sqrt{\int_{-\pi}^{\pi} \frac{\cos^2 nx}{\pi} dx} = 1$$

2. Fourier Series

1807 presented by Fourier application:

$$\begin{cases} Acoustics \\ optics \\ ther\ \mathrm{mod}\ ynamics \\ othere \end{cases}$$

(1)Set of functions

$\{1, \cos\frac{n\pi}{P}x, \sin\frac{n\pi}{P}x\}$ n,m=1,2,3…

is orthogonal on [-P,P]

$$\int_{-P}^{P} \cos\frac{n\pi}{P}x \sin\frac{m\pi}{P}xdx = 0 \qquad m \neq n$$

$$\int_{-P}^{P} \cos\frac{n\pi}{P}x \cos\frac{m\pi}{P}xdx = 0 \qquad m \neq n$$

$$\int_{-P}^{P} \sin\frac{n\pi}{P}x \sin\frac{m\pi}{P}xdx = 0$$

$$\int_{-P}^{P} 1 \cdot \cos\frac{n\pi}{P}xdx = 0$$

$$\int_{-P}^{P} 1 \cdot \sin\frac{n\pi}{P}xdx = 0$$

(2)Fourier Series

Function $f(x)$ defined on [-P,P] that can be expanded in the trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x)...(1)$$

Where

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(x)dx$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x)\cos\frac{n\pi}{P}xdx$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x)\sin\frac{n\pi}{P}xdx$$

a_0, a_n, b_n :

Fourier coefficients

Multiplying Eq(1) by 1 and integrating from − P to P, we have

$$\int_{-P}^{P} f(x)dx = \int_{-P}^{P} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \int_{-P}^{P} a_n \cos\frac{n\pi}{P} xdx + \int_{-P}^{P} b_n \sin\frac{n\pi}{P} xdx = a_0 P$$

$$\therefore a_0 = \frac{1}{P}\int_{-P}^{P} f(x)dx$$

Multiply Eq(1) by $\frac{m\pi}{P}x$ as and integrate from − P to P to yield

$$\int_{-P}^{P} f(x)\cos\frac{m\pi}{P} xdx = \int_{-P}^{P} \frac{a_0}{2}\cos\frac{m\pi}{P} xdx + \sum_{n=1}^{\infty}\left\{a_n \int_{-P}^{P} \cos\frac{n\pi}{P} x\cos\frac{m\pi}{P} xdx + b_n \int_{-P}^{P} \cos\frac{n\pi}{P} x\sin\frac{n\pi}{P} xdx\right\}$$

Here

$$\int_{-P}^{P} \cos\frac{m\pi}{P} x\sin\frac{n\pi}{P} xdx = 0 \qquad \text{orthogonal}$$

$$\int_{-P}^{P} \cos\frac{m\pi}{P} x\cos\frac{n\pi}{P} xdx = \begin{cases} 0 \\ \left\|\cos\frac{n\pi}{P} x\right\|^2 = P(m=n) \end{cases}$$

$$\therefore \int_{-P}^{P} f(x)\cos\frac{n\pi}{P} xdx = a_n \int_{-P}^{P} \cos^2\frac{n\pi}{P} xdx = a_n P$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x)\cos\frac{n\pi}{P} xdx$$

Similarly, multiply Eq(1) by $\sin\frac{m\pi}{P}x$ and integrate to obtain

$$\int_{-P}^{P} f(x)\sin\frac{n\pi}{P} xdx = b_n \int_{-P}^{P} \sin^2\frac{n\pi}{P} xdx = b_n P$$

$$\therefore b_n = \frac{1}{P}\int_{-P}^{P} f(x)\sin\frac{n\pi}{P} xdx$$

(3) Convergence of a Fourier Series

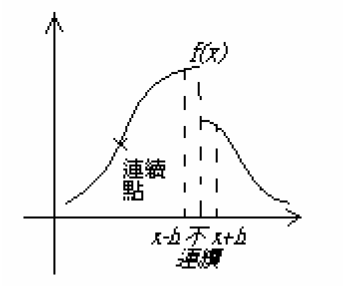
Let f(x) and f `(x) be piecewise continuous on [-P, P] then

A. Fourier Series of f(x) on [-P, P] Converges to f(x) at a point of continuity.

B. At a point of discontinuity, the Fourier Series Converges to the average

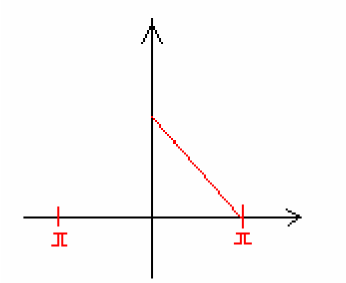
$$\frac{f(x+)+f(x-)}{2} \qquad \text{Here } f(x+)=\lim_{h\to 0} f(x+h) \quad \text{right limit}$$

$$f(x-)=\lim_{h\to 0} f(x-h) \quad \text{left limit}$$



Ex. Expand $f(x)=\begin{cases}0 & -\pi<x<0\\ \pi-x & 0<x<\pi\end{cases}$ in a Fourier Series

Sol:



$p=\pi$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi}{P}x+b_n\sin\frac{n\pi}{P}x\right]=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)=\frac{1}{\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}(\pi-x)dx\right]=\frac{\pi}{2}$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nx\,dx=\frac{1-\cos n\pi}{n^2\pi}=\frac{1-(-1)^n}{n^2\pi} \qquad n=1,2,3,\ldots$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx=\frac{1}{n}$$

$$\therefore\ f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{1}{n}\sin nx\right)$$

at x=0 Fourier series converges to $\dfrac{f(x+)+f(x-)}{2}=\dfrac{\pi+0}{2}=\dfrac{\pi}{2}$

$$\therefore \frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\right)=\frac{\pi}{2}$$

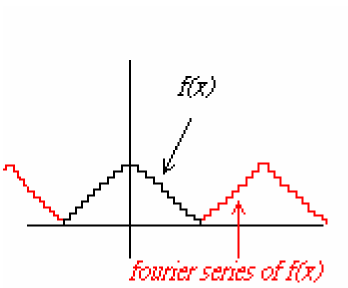
$$\begin{aligned}\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\right)&=\frac{\pi}{4}+\left(\frac{2}{\pi}+\frac{2}{3^2\pi}+\frac{2}{5^2\pi}+\ldots\right)\\&=\frac{\pi}{4}+\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)\\&=\frac{\pi}{2}\end{aligned}$$

$$\therefore \boxed{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}}$$

(4)periodic extension

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right) \qquad \ldots(1)$$

The right hand side of Eq.(1) is periodic with period 2p.
∴Fourier series represents $f(x)$ on [-p, p] and gives periodic extension of $f(x)$ outside [-p, p].



Therefore, we may assume at the beginning that f(x) is periodic with period 2p, namely,
$f(x+2p)=f(x)$
If $f(x)$ is a periodic function of period $2\pi(p=\pi)$, then Fourier series becomes,

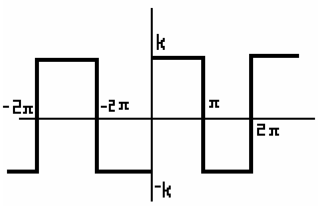
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx$$

Ex.

$$f(x)=\begin{cases}-k & -\pi<x<0\\ k & 0<x<\pi\end{cases}$$

$$f(x+2\pi)=f(x)$$



Fourier series of $f(x)$?

Sol. $p=\pi$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}[\int_{-\pi}^{0}-kdx+\int_{0}^{\pi}kdx]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}[\int_{-\pi}^{0}-k\cos nxdx+\int_{0}^{\pi}k\cos nxdx]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}[\int_{-\pi}^{0}-k\sin nxdx+\int_{0}^{\pi}k\sin nxdx]=\frac{2k}{n\pi}(1-\cos nx)$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\frac{2k}{n\pi}(1-\cos nx)\sin nx$$

$$=\frac{4k}{n\pi}(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...)$$

at x=0 Fourier series converges to $\frac{f(x+)+f(x-)}{2}=\frac{k+(-k)}{2}=0$

$$f(0)=\sum_{n=1}^{\infty}\frac{4k}{n\pi}(1-\cos n\pi)\sin n\pi\,|_{x=0}=0$$

at $x=\frac{\pi}{2}$ (連續點)

$$f\left(\frac{\pi}{2}\right)=k=\sum_{n=1}^{\infty}\frac{4k}{n\pi}(1-\cos n\pi)\sin n\pi\Big|_{x=\pi/2}$$

$$=\frac{4k}{\pi}(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots)|_{x=\pi/2}$$

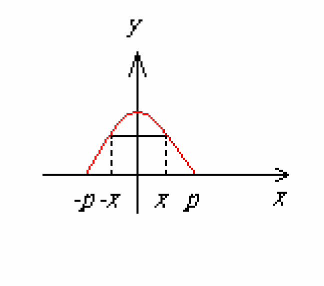
$$=\frac{4k}{\pi}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\cdots)$$

$\therefore 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\cdots=\frac{\pi}{4}\cdots\cdots$ Fourier series at specific point

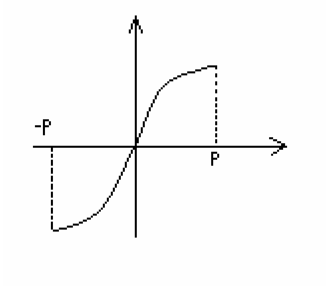
3. Fourier Cosine & Sine Series

(1) Even and Odd function:

A. Even function



B. Odd function



(2) Properties:

A. If f(x) is even, then $\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)dx$

B. If f(x) is odd, then $\int_{-p}^{p} f(x)dx = 0$

(3) Fourier Cosine Series:

Let f(x) be even function on [-p, p] Fourier series of even function f(x).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x]$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx = 0$$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p} \cdots\cdots$ Fourier cosine series

(4) Fourier sine series:

Let f(x) be odd function on [-p, p], Fourier Coefficients for odd function are

$$a_0 = 0$$

$$a_n = 0$$

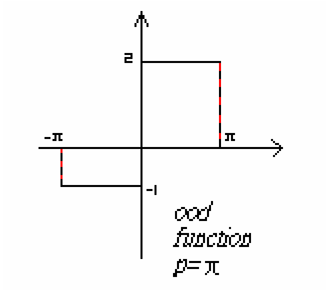
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

$\therefore$ Fourier series for odd function f(x) is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p} x \cdots\cdots \text{Fourier sine series}$$

Ex: Expand f(x)= $\begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$ **in Fourier series:**

Sol:



$$\therefore f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (1)\sin nx dx$$

$$= \frac{2}{\pi}\ \frac{-\cos n\pi}{n}\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\ \frac{1-\cos n\pi}{n}$$

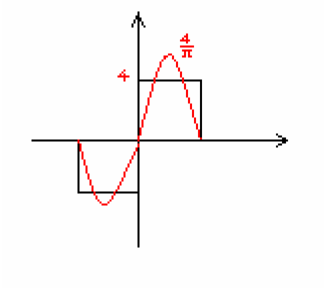
$$= \frac{2}{\pi}\ \frac{1-(-1)^n}{n}$$

$$= \begin{cases} \dfrac{4}{\mathrm{n}\pi} & \mathrm{n} = 1,3,5,7\cdots\cdots \\ 0 & \mathrm{n} = 2,4,6,8\cdots\cdots \end{cases}$$

$$\therefore f(x)=\sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin nx$$

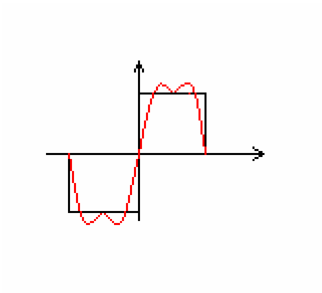
$$= \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\cdot)$$

If n=1



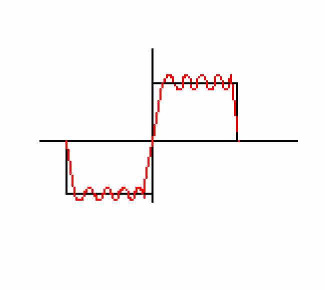
$$f(x) = \frac{4}{\pi}\sin x$$

If n=2



$$f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$$

If n=↑



The behavior of a Fourier series near a discontinuous point of f(x) is known as Gibb Phenomenon.

(5) Sum of Functions:

A. Fourier coefficients of f(x) + g(x)

=Sums of the corresponding Fourier coefficients of f(x) and g(x).

B. Fourier coefficients of $cf(x)$

=C times the Fourier coefficients of f(x).

Ex: $\begin{cases} f(x) = \pi + x & -\pi \le x \le \pi \\ f(x+2\pi) = f(x) \end{cases}$ **Fourier series?**

Sol:

$$f(x) = f_1(x) + f_2(x)$$
$$= x + \pi$$

$$f_1(x) = x$$

$$f_2(x) = \pi$$

$$f_1(x) = \frac{a_{01}}{2} + \sum_{n=1}^{\infty} [a_{n1} \cos nx + b_{n1} \sin nx]$$

$$f_2(x) = \frac{a_{02}}{2} + \sum_{n=1}^{\infty} [a_{n2} \cos nx + b_{n2} \sin nx]$$